Physics of Black Holes

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Preface

One of the most exciting predictions of Einstein's theory of gravitation is that there may exist 'black holes': putative objects whose gravitational fields are so strong that no physical bodies and signals can break free of their pull and escape. Even though a completely reliable discovery of a black hole has not yet been made, several objects among those scrutinized by astrophysicists will very likely be conformed as black holes. The proof that they do exist, and an analysis of their properties, would have a significance going far beyond astrophysics. Indeed, what is involved is not just the discovery of yet another, even if extremely remarkable, astrophysical object, but a test of the correctness of our understanding the properties of space and time in extremely strong gravitational fields.

Theoretical research into the properties of black holes and into the possible corollaries of the hypothesis that they exist, has been carried out with special vigor since the beginning of the 1970s. In addition to those specific features of black holes that are important for the interpretation of their possible astrophysical manifestations, the theory has revealed a number of unexpected characteristics of physical interactions involving black holes.

By now, a fairly detailed understanding has been achieved of the properties of the black holes, their possible astrophysical manifestations, and the specifics of the various physical processes involved. Furthermore, profound links were found between black-hole theory and such seemingly very distant fields as thermodynamics, information theory, and quantum theory. The branch of physics that is now referred to as black-hole physics was born and actually took shape as a full-blooded scientific discipline during the past two decades at the junction of the theory of gravitation, astrophysics, and classical and quantum field theories.

This book is written to introduce the reader to the physics of black holes and the methods employed in it, and to the main results of this relatively young and rapidly progressing branch of physics. The main attention is focused on the questions that were answered relatively recently and, thus, could not be adequately reflected in earlier textbooks and reviews.[1*] Those aspects that are relatively familiar will be presented briefly (but as clearly as possible).

* Notes to the text are to be found at the end of the book, following the Appendix.

We have tried to make the representation lucid, not only for a specialist, but also for a broad spectrum of physicists and astrophysicists who do not have a special knowledge of black-hole physics.

An attempt is made to explain, first and foremost, the physical essentials of the phenomena; and only after this do we pass on to the mathematical means of describing them. These objectives decided both the spectrum of selected topics and the style of presentation. A conscious attempt has been made to avoid the excessive rigorousness in the formulations and proofs of theorems on black holes. Quite often, only the principal idea is given instead of a complete proof, then the successive stages of the proof are outlined, and references to the original papers are supplied where the reader will find the required details. This approach was chosen not only because the excellent monographs of Penrose (1968), Hawking and Ellis (1973), and Chandrasekhar (1983) cover most of the material omitted in this book; but also because we are of the opinion that excessive rigorousness stands in the way of understanding the physical ideas that are at the foundation of the specific properties of black holes.

(After this monograph had been completed and sent to NAUKA editors, an excellent book appeared, written by Thorne, Price, and MacDonald (1986), which treats a number of physical aspects of the black-hole problem.)

The astrophysical aspects of the theory of black holes had to be virtually abandoned. Volume limitations did not allow us to include any satisfactorily complete review of this field into the present monograph. The relevant topics may well deserve a separate volume. Again, analysis is replaced with references to original publications.

We wish to express our gratitude to I.G. Virko for his help in preparing the manuscript of the Russian edition for publication.

Moscow *I.D. Novikov*
September 1987 *V.P. Frolov*

Preface to the English Edition

The English translation of our book appears three years after the first Russian edition of 1986. These were years of further progress in the physics of black holes. Correspondingly, we made an attempt to include into the present edition the most important new results. However, we preserved the original spirit of the book in not striving to encompass the *entire* recent literature on the subject. Nevertheless, we hope that the English edition presents the current situation in black-hole physics. We also used this opportunity for correcting misprints and some imperfections that rereading detected in the Russian edition. We are grateful to the translator of the book, Dr Vitaly Kisin, for his valuable help in preparing the English edition.

Moscow — *Igor Novikov*
October 1988 — *Valery Frolov*

Chapter 1

Introduction

A *black hole* is, by definition, a region in spacetime in which the gravitational field is so strong that it precludes even light from escaping to infinity.

A black hole is formed when a body of mass M contracts to a size less then the so-called *gravitational radius* $r_g = 2GM/c^2$ (G is Newton's gravitational constant and c is the speed of light). The velocity required to leave the boundary of the black hole and move away to infinity (the escape velocity) equals the speed of light. One easily concludes then that neither signals nor particles can escape from the region inside the black body, since the speed of light is the limiting propagation velocity for physical signals. This conclusion is of absolute nature in Einstein's classical theory of gravitation because the gravitational interaction is universal. The role of gravitational charge is played by mass whose value is proportional to the total energy of the system. Hence, all objects with nonzero energy participate in the gravitational action.

Einstein's theory of gravitation, alias general relativity (GR), is employed to the full in the description of black holes.[1] It may appear at first glance that one cannot hope to obtain an acceptably full description of black holes, owing to the complexity of the equations involved and, among other factors, their essential nonlinearity. Fortunately, it was found that shortly after its formation, any black hole becomes stationary and its field is determined in a unique manner by a small number of parameters, namely, its mass and angular momentum, and its electric charge (if it is charged). The physical reason for this striking property of black holes is the fact that in the extremely strong field of a black hole, only very special type of configuration of physical fields (including the gravitational field) can be stationary.

Since signals cannot escape from a black hole while physical objects and radiation can fall into it, the surface bounding the black hole in spacetime (called the *event horizon*) is a light-like surface. The birth of a black hole signifies the formation of a nontrivial causal structure in spacetime. As a result of these specific features, new methods had to be developed to analyze the interaction of black holes with physical fields and matter, and with other black holes.

The term 'black hole' was introduced by Wheeler in 1968, although the possibility of such objects had been discussed by Michell and Laplace within the framework of the Newtonian theory at the end of the 18th century [see Barrow and

Silk (1983), Novikov (1985)]. In general relativity, the problem arose, in fact, within a year after the theory had been developed, i.e., after Schwarzschild (1916) obtained the first exact (spherically symmetric) solution of Einstein's equations in vacuum. In addition to a singularity at the center of symmetry (at $r = 0$), this solution had an additional singularity on the gravitational-radius surface (at $r = r_g$). More than a third of a century elapsed before profound understanding of the structure of spacetime in strong gravitational fields was achieved as a result of analysis of the 'unexpected' features of the Schwarzschild solution by Flamm (1916), Weyl (1917), Eddington (1924), Lemaitre (1933), Einstein and Rosen (1935), and the complete solution of the formulated problem was obtained [Synge (1950), Finkelstein (1958), Fronsdal (1959), Kruskal (1960), Szekeres (1960), Novikov (1963, 1964a)]. The length of this interval may have been influenced by the general belief that nature could not admit a body whose size would be comparable to its gravitational radius; this viewpoint was shared by the creator of general relativity himself. However, the interest to the properties of supercontracted gravitational systems was greatly enhanced in the Thirties after the work of Landau, Baade, Zwicky, Oppenheimer and Volkoff who showed that neutron stars are possible, with a radius only several times that of the gravitational radius. Gravitational collapse of a massive star which produces a black hole was first described by Oppenheimer and Snyder (1939).

The next period began in the middle sixties when intensive theoretical studies were initiated on the general properties of black holes and their classical interactions, after the work of Synge, Kruskal and others who obtained the complete solution for the Schwarzschild problem, and of Kerr (1963) who discovered a solution describing the gravitational field of a rotating black hole. The now classical theorems stating that 'black holes have no hair' (that is, no external individual attributes except mass, angular momentum, and charge), that a black hole contains a singularity, and that the black hole area cannot decrease, were proved during this period. These and other results made it possible to reconstruct the qualitative picture of the formation of a black hole, its possible further evolution, and its interaction with matter and classical physical fields. Many of these results were summarized in the well-known monographs of Misner, Thorne, and Wheeler (1973) and Hawking and Ellis (1973).

After pulsars (neutron stars) were discovered at the end of the Sixties, astrophysicists had to face the possibility of observational detection of black holes. The analysis of the accretion of matter on lone black holes and on those in binary systems predicted that accreting black holes may constitute powerful sources of X-rays [Novikov and Zel'dovich (1966), Shklovsky (1967), Burbidge (1972)]. The progress of X-ray astronomy and the studies on X-ray satellites that began in the 1970s led to the discovery of a number of X-ray sources, one of which, located in the Cygnus constellation (Cyg X-1), is likely to be a black hole.

Nearly 15 years of constantly studying this object provided confirmation of the initial hypothesis. Several more 'candidates' to the black hole family are known at present. There is also a good reason to believe that the nuclei of active galaxies (and possibly of any galaxy) and quasars contain massive or supermassive black holes [see Blandford and Thorn (1979), Rees (1982)].

The discussion of the possible observational aspects of black-hole study drew considerable attention to the problem of the motion of particles and physical fields in the spacetime of stationary black holes.

By now, this problem, which is predominantly mathematical and involves the integration of the equations of geodesics and the construction of the expansion in the eigenfunctions of the invariant wave operators in the Kerr metric, has been generally solved. Numerous relevant results are summarized in the recently published monograph by Chandrasekhar (1983), 'The mathematical theory of black holes'.

The sensational 'news' of the possible discovery of a black hole did not die down when a new unexpected result obtained by Hawking (1974, 1975) again focussed physicists' attention on black holes. It was found that as a result of the instability of the vacuum in the strong gravitational field of a black hole, these objects are sources of radiation. If the black-hole mass was small, it would decay over a time shorter than the age of the Universe. Such small black holes, now called primordial black holes, may have been formed only at a very early stage of the Universe's evolution [Zel'dovich and Novikov (1966, 1967), Hawking (1971a)]. In principle, the discovery of primordial black holes or of their decay products would supply valuable information on the physical processes occurring in the Universe at that period.

Hawking's discovery stimulated a large number of papers which analyzed specific features of quantum effects in black holes. In addition to a detailed description of the effects due to the creation of real particles escaping to infinity, substantial progress has been achieved in recent years in the understanding of the effect of vacuum polarization in the vicinity of a black hole. This effect is important for the construction of a complete quantum description of an 'evaporating' black hole.

This book presents a systematic exposition of black-hole physics and is organized as follows.

In order to descriptively introduce certain important notions and to place emphasis on the fundamental problems, the authors have attempted to make the presentation in the opening chapters particularly simple and clear. This is especially true of Chapter 2 which describes the properties of the simplest spherical black hole. This chapter also presents the properties of spacetime within a black hole.

Chapter 3 treats the propagation of weak physical fields near black holes. The main attention is focussed on the evolution of weak gravitational fields. This is especially important in the problem of the stability of a black hole under external perturbations, and in the problem of the emission of gravitational waves by bodies (and fields) moving in the neighborhood of a black hole. Here, we also treat the creation of a black hole in the collapse of a slightly nonspherical body.

Chapter 4 introduces the most important properties of rotating black holes and of those having a nonzero electric charge.

Chapter 5 presents the general theory of nonstationary black holes and the results of the existence of singularities.

Chapter 6 gives the proof to the uniqueness theorem for stationary black holes.

Chapter 7 treats the methods of the analysis of electromagnetic fields close to a stationary black hole, using the '3+1' decomposition of spacetime.

Chapter 8 discusses various physical effects in the field of a black hole: superradiance, shift in the self-energy of charged particles, mutual transformation of electromagnetic and gravity waves, the motion and deformation of black holes in an external field, and interactions between black holes.

Chapters 9 and 10 discuss the quantum physics of black holes. Chapter 9 gives the general solution to the problem of creation of particles in the field of a stationary black hole. Chapter 10 summarizes the results on the vacuum polarization in the neighborhood of a black hole.

Chapter 11 discusses the thermodynamic analogy in black-hole physics.

Chapter 12 treats various aspects of the structure of spacetime inside black holes.

Chapter 13 presents the results concerning primordial black holes, the theory of white holes and semiclosed worlds, and a possible role of elementary black holes in quantum gravity.

The book ends with an Appendix containing selected results from Riemannian geometry and general relativity, and lists the important formulas employed in the main text.

The signs in the definitions of ds^2, curvature tensor, and Ricci tensor are chosen as in the monograph of Misner, Thorne, and Wheeler (1973). In Chapters 2 and 3, all formulas are written with dimensional physical constants c and G. Beginning with Chapter 4, where more complicated matter is treated and the use of dimensional constants would yield unwieldy expressions, we employ everywhere (except in the final formulas or where specified otherwise) the system of units $c = G = \hbar = k = 1$.

Chapter 2

A Spherically Symmetric Black Hole

2.1. Spherically Symmetric Gravitational Field

We begin the analysis of the physical properties of black holes with the simplest case in which both the black hole and its gravitational field are spherically symmetric.

The spherically symmetric gravitational field (spacetime with spherical three-dimensional space) is described in every textbook on general relativity [see, e.g., Landau and Lifshitz (1973), Misner, Thorne, and Wheeler (1973)]. Therefore, here we will only reproduce the necessary results. Let us write the expression for a squared interval far from strong gravitational fields (i.e., where special relativity is valid), using the spherical spatial coordinate system (r, θ, φ):

$$ds^2 = -c^2 dt^2 + dl^2 = -c^2 dt^2 + dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2), \tag{2.1.1}$$

where c is the speed of light, and dl is the distance in three-dimensional space.

Now let us consider a curved spacetime but preserve the condition of spatial spherical symmetry. Spacetime is not necessarily empty, it may contain matter and physical fields (which are, of course, also spherically symmetric if their gravitation is considered). Calculations show [see, e.g., Misner, Thorne, and Wheeler (1973)] that in this case the interval can always be written (after a suitable coordinate system has been chosen) in the form[1]

$$ds^2 = g_{00}(x^0, x^1)\, dx^{0^2} + g_{11}(x^0, x^1)\, dx^{1^2} + (x^1)^2 (d\theta^2 + \sin^2\theta \, d\varphi^2). \tag{2.1.2}$$

The same components of the metric tensor are nonzero in (2.1.2) as in the expression for flat space, (2.1.1). The components g_{00} and g_{11} are functions only of x^0 and x^1 and are independent of θ and φ.

The coordinates in which the expression for g_{22} is written in the form $(x^1)^2$ are called *curvature coordinates*. Usually the x^1 coordinate is denoted by r [by analogy to (2.1.1)] and $x^0/c \equiv t$. We will see that this choice of symbols is not always logically justified inside a black hole (see Section 2.4).

If a spherical gravitational field is considered not in vacuum, then in the general case matter moves radially in the three-dimensional coordinate system defined by the coordinates x^1, θ, φ, that is, energy flows exist. Sometimes it is more convenient

to choose a different frame of reference, for example, a comoving one, but which is also spherically symmetric. All such reference frames possess the following property. The points of which one such reference frame is composed move radially with respect to some other frame. The relation between different reference frames which preserve the spherical symmetry is given by the transformations

$$\tilde{x}^0 = \tilde{x}^0(x^0, x^1), \tag{2.1.3}$$

$$\tilde{x}^1 = \tilde{x}^1(x^0, x^1), \tag{2.1.4}$$

$$\tilde{\theta} = \theta, \quad \tilde{\varphi} = \varphi. \tag{2.1.5}$$

The coordinates in the new reference frame are marked with a tilde. Expression (2.1.4) describes the radial motion of the points of the new reference frame (its coordinates are $\tilde{x}^1$ = const) with respect to the older one. Once (2.1.4) has been chosen, thus defining the new frame of reference, it is always possible to choose (2.1.3), which defines the time coordinate in the new system, in such a way that the component $\tilde{g}_{01}$ would not arise and the general expression for ds^2 would have the form

$$ds^2 = \tilde{g}_{00}(\tilde{x}^0, \tilde{x}^1)\, d\tilde{x}^{0^2} + \tilde{g}_{11}(\tilde{x}^0, \tilde{x}^1)\, d\tilde{x}^{1^2} +$$

$$+ \tilde{g}_{22}(\tilde{x}^0, \tilde{x}^1)(d\tilde{\theta}^2 + \sin^2\tilde{\theta}\, d\tilde{\varphi}^2). \tag{2.1.6}$$

Note that the expression for g_{22} can be written in the form

$$\sqrt{\tilde{g}_{22}} = x^1(\tilde{x}^0, \tilde{x}^1), \tag{2.1.7}$$

where $x^1 = x^1(\tilde{x}^0, \tilde{x}^1)$ is the solution of (2.1.3) and (2.1.4) for x^1. It describes the radial motion of the points of the older reference frame (with the coordinates x^1 = const) with respect to the new one.

2.2. Spherically Symmetric Gravitational Field in Vacuum

Consider a spherical gravitational field in vacuum. The solution to Einstein's equations for this case were found by Schwarzschild (1916); it has the following form [see Landau and Lifshitz (1973)]:

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{2.2.1}$$

where G is Newton's gravitational constant and M is the mass of the field source.

A most important property of this solution is that it is independent of the temporal coordinate t and depends only on r, and that it is determined by a single parameter M, that is, the total mass of the gravitational source which produces the field. Even if the field source involves radial motions (which preserve spherical symmetry), the field beyond the region occupied by matter remains constant [this statement is known as *Birkhoff's theorem* (1923)]. Far from the center of gravity (as $r \to \infty$), spacetime converts to the flat Minkowski spacetime with metric (2.1.1). The coordinates t, r, θ, ϕ in which (2.1.1) is written are called the *Schwarzschild coordinates*, and the frame of reference that they form is called the *Schwarzschild reference frame*. For ordinary measurement of length in a small neighborhood of each spatial point, we can use a local Cartesian coordinate system (x, y, z):

$$\delta x = \sqrt{g_{11}}\, dr = \left(1 - \frac{2GM}{c^2 r}\right)^{-1/2} dr, \tag{2.2.2}$$

$$\delta y = \sqrt{g_{22}}\, d\theta = r\, d\theta, \tag{2.2.3}$$

$$\delta z = \sqrt{g_{33}}\, d\phi = r \sin\theta\, d\phi. \tag{2.2.4}$$

The factor $(1 - 2GM/c^2 r)^{-1/2}$ in (2.2.2) reflects the curvature of the three-dimensional space.

The physical time τ at a given point r is given by the expression

$$d\tau = \frac{\sqrt{-g_{00}}}{c}\, dx^0 = \sqrt{-g_{00}}\, dt = \left(1 - \frac{2GM}{c^2 r}\right)^{1/2} dt. \tag{2.2.5}$$

Far from the gravitational center (as $r \to \infty$), we have $d\tau = dt$, that is, t is the physical time of the observer removed to infinity.

At smaller r, the time τ runs progressively slower in comparison with the time t at infinity. As $r \to 2GM/c^2$, we find $d\tau \to 0$.

Let us now calculate the acceleration of free fall of a body which is at rest (or moves at a low velocity $v \ll c$) in the Schwarzschild reference frame. Using formula (A.63) (see Appendix), we find

$$F = \sqrt{F_i F^i} = \frac{GM}{r^2 (1 - 2GM/c^2 r)^{1/2}}. \tag{2.2.6}$$

The acceleration points along the radius. As $r \to 2GM/c^2$, acceleration tends to infinity. The singularity in time flux arising as $r \to 2GM/c^2$ [see (2.2.5)] and that in the expression for acceleration F [see (2.2.6)] demonstrate that at this value of r, the Schwarzschild reference frame has a physical singularity.[2] The quantity $r = r_g = 2GM/c^2$ is called the *Schwarzschild radius* (or gravitational radius; see p. 1), and the sphere of radius r_g is said to be the *Schwarzschild sphere*. We will later give a

detailed analysis of the physical meaning of the singularity at $r = r_g$. The following aspects are emphasized here.

The Schwarzschild reference frame is static and nondeformable [$g_{\alpha\beta}$ is independent of t, $g_{0i} = 0$, $D_{ik} = 0$; see (A.60)]. It can be thought of as a coordinate lattice 'welded' of weightless rigid rods which fill the space around the black hole. We can study the motion of particles relative to this lattice, the evolution of physical fields at different points of this lattice, and so on. The Schwarzschild lattice thus, to some extent, resembles the lattice of fixed coordinates in the invariable Newtonian space of nonrelativistic physics. Of course, the geometry of the three-dimensional Schwarzschild space around a gravitational center is non-Euclidean, in contrast to the Euclidean Newtonian space of nonrelativistic physics. In other respects the properties of these frames are very similar.[3] This factor is a great help in our intuition.

When we speak of the motion of particles in the Schwarzschild field, we mean the motion and evolution of fields in this analogue of absolute Newtonian space.[4] As a result of the presence of the critical radius, $r_g = 2GM/c^2$, in the spherical field in vacuum, where the free-fall acceleration becomes infinite, such a rigid nondeformable lattice cannot be extended to $r \leq r_g$, since this region contains no nondeformable space (no analogue of the Newtonian space). The fact that F tends to infinity at r_g is an indication that at $r \leq r_g$ all systems must be nonrigid in the sense that $g_{\alpha\beta}$ must be a function of time and all systems must be deformed (all bodies must fall centerward). We will see that this is indeed the case.

Note that these specificities at $r = r_g$ do not indicate that a singularity of the type of infinite curvature, or something similar, exists in the geometry of the four-dimensional spacetime. We shall see later that here the spacetime is quite regular, and the singularity at r_g points to a physical singularity only in the Schwarzschild reference frame, that is, it signifies the impossibility of extending this reference frame as a rigid and nondeformable one (not falling on the center) to $r \leq r_g$.

Note in conclusion that r_g is extremely small, even for heavenly bodies. Thus, $r_g = 0.9$ cm for the Earth's mass and $r_g = 3$ km for the Sun's mass. If $r \gg r_g$, the Schwarzschild field is the ordinary Newtonian gravitational field with the free fall acceleration $F = GM/r^2$ and negligible curvature of the three-dimensional space. Outside the typical heavenly bodies (and all ordinary bodies as well), the gravitational field is the Newtonian field, because their sizes are typically much greater than r_g (The only known exceptions are neutron stars and black holes.) The Schwarzschild solution within these bodies is not valid, and, obviously, the gravitational field is also Newtonian, with enormously high accuracy.

We will see later that a spherical black hole is formed when a nonrotating spherical body collapses to a size below its gravitational radius. But before we discuss this process of the birth of a black hole, we need to look at the laws of radial motion of test particles in the Schwarzschild field.

2.3. Radial Motion of Test Particles in the Schwarzschild Field

We begin with the radial motion of the photon which always propagates at the fundamental velocity c. This is also true for any other ultrarelativistic particle. For this particle, $ds = 0$. For the radial motion, $d\theta = d\phi = 0$. Substituting $ds = d\theta = d\phi = 0$ into (2.2.1), we find the equation of motion

$$\frac{dr}{dt} = \pm c\left(1 - \frac{r_g}{r}\right). \tag{2.3.1}$$

Recall that dr/dt is the rate at which the coordinate r changes with the time t of a distant observer (and not with respect to the physical time τ at a given point), that is, this is the coordinate (not the physical) velocity. The physical velocity is the rate of change of physical distance, dx [see (2.2.2)], in the physical time τ [see (2.2.5)]

$$\frac{dx}{d\tau} = \pm \frac{\sqrt{g_{11}}\, dr}{\sqrt{-g_{00}}\, dt} = \pm c\,.$$

Of course, the physical velocity of the photon (in any reference frame) is always equal to c.

From the standpoint of a distant observer (and according to his clock), the change dx in the physical radial distance with t is

$$\frac{dx}{dt} = \pm c\left(1 - \frac{r_g}{r}\right)^{1/2}. \tag{2.3.2}$$

Therefore, a distant observer finds that a light ray close to r_g moves slower, and as $r \to r_g$, $dx/dt \to 0$. Obviously, this behavior reflects the slowing down of time close to r_g [see (2.2.5)].

What time does a photon take, by the clock of a distant observer, in order to reach the point r_g, if the motion starts radially from $r = r_1$? We integrate Equation (2.3.1) and obtain

$$t = \frac{r_1 - r}{c} + \frac{r_g}{c}\ln\left(\frac{r_1 - r_g}{r - r_g}\right) + t_0\,, \tag{2.3.3}$$

where r_1 is the position occupied by the photon at the moment t_0. Expression (2.3.3) shows that $t \to \infty$, as $r \to r_g$. Whatever the coordinate r_1 from which the photon starts its fall, by the clock of the distant observer, the time t taken by the photon to reach r_g is infinite.

How does the photon energy change in the course of the radial motion? Energy is proportional to frequency. Let us look at the evolution of frequency. Suppose that light flashes at a point $r = r_1$ at an interval Δt. The field being static, the flashes

will reach the observer at $r = r_2$ after the same interval Δt. The ratio of the proper time intervals at these two points is

$$\frac{\Delta\tau_1}{\Delta\tau_2}=\frac{\sqrt{-g_{00}(r_1)}\ \Delta t}{\sqrt{-g_{00}(r_2)}\ \Delta t}\ ;$$

hence, the ratio of frequencies is

$$\frac{\omega_1}{\omega_2}=\frac{\Delta\tau_2}{\Delta\tau_1}=\sqrt{\frac{g_{00}(r_2)}{g_{00}(r_1)}}=\sqrt{\frac{1-r_g/r_2}{1-r_g/r_1}}\ . \tag{2.3.4}$$

The frequency of a quantum decreases as it leaves the gravitational field and increases as it moves centerward. This effect is called the red and blue gravitational shift, respectively.

Let us now look at the radial motion of nonrelativistic particles in vacuum. We begin with free motion in which no nongravitational forces act on a particle (free fall, motion along a geodesic). The integration of the equation for a geodesic in the case $d\theta = d\phi = 0$ [see Bogorodsky (1962)] yields the expression

$$\frac{dr}{dt}=\pm\frac{(1-r_g/r)\,[(E/mc^2)^2-1+r_g/r]^{1/2}}{E/mc^2}\,c\ , \tag{2.3.5}$$

where E is the constant of motion describing the total energy of a particle, including its mass m. If the particle is at rest at infinity where the gravitational field vanishes, then $E = mc^2$. In the general case, the value E/mc^2 may be greater or smaller than unity, but E for a particle moving outside the sphere of radius r_g is invariably positive.

At a large distance $r \gg r_g$, we find that for nonrelativistic particles $|(E - mc^2)/mc^2| \ll 1$, and expression (2.3.5) is rewritten in the form

$$\frac{m\,(dr/dt)^2}{2}=(E-mc^2)+\frac{GmM}{r}\ . \tag{2.3.6}$$

The quantity $\mathcal{E} = E - mc^2$ is the energy of a particle in Newtonian theory (where the rest, or proper, mass is not included in the energy), and thus expression (2.3.6) converts to the energy conservation law in Newtonian theory.

Recall again that dr/dt in (2.3.5) is the coordinate (not physical) velocity. The physical velocity measured by an observer which is at rest in the Schwarzschild reference frame situated in the neighborhood of the freely moving body is

$$\frac{dx}{d\tau}=\sqrt{\frac{g_{11}}{|g_{00}|}}\ \frac{dr}{dt}=\pm\frac{[(E/mc^2)^2-1+r_g/r]^{1/2}}{E/mc^2}\,c\ . \tag{2.3.7}$$

If the falling body approaches r_g, the physical velocity constantly increases: $dx/d\tau \to c$, as $r \to r_g$. By the clock of the distant observer, the velocity dx/dt tends to

zero as $r \to r_g$, as in the case of the photon. This fact reflects the slowing down of time as $r \to r_g$.

What is the time required for a body falling from a point $r = r_1$ to reach the gravitational radius r_g (by the clock of the distant observer)?

The time of motion from r_1 to r_g is given by the integral of (2.3.5). This integral diverges as $r \to r_g$. This result is not surprising because $\Delta t \to \infty$ as $r \to r_g$ even for light, and nothing is allowed to move faster than light. Furthermore, the divergence of Δt for a falling body is of the same type as for light because the velocity of the body, v, always tends to c as $r \to r_g$. Obviously, whatever the force acting on a particle, the time Δt of reaching r_g is always infinite, because in this case again $v < c$. We conclude that both the free fall and motion towards r_g at any acceleration always take infinite time measured by the clock of the distant observer.

Let us return to a freely moving particle. What is the time ΔT for reaching r_g measured by the clock of the falling particle itself? It is found from the formula

$$\Delta T = \frac{1}{c}\int_{r_g}^{r_1} |\,\mathrm{d}s\,|\,,$$

where ds is taken along the world line of the particle. Here, using the expression for ds from (2.2.1), for $\mathrm{d}\theta = \mathrm{d}\phi = 0$, we find

$$\Delta T = \frac{1}{c}\int_{r_g}^{r_1} \sqrt{\left|\frac{g_{00}}{(\mathrm{d}r/c\,\mathrm{d}t)^2} + g_{11}\right|}\,\mathrm{d}r. \tag{2.3.8}$$

In order to calculate ΔT, we substitute into (2.3.8) the expression for dr/dt from (2.3.5). It is easy to show that the integral converges and the interval ΔT is finite. In the particular case of $E = mc^2$, when the particle falls at the parabolic (escape) velocity (i.e., dr/dt = 0 at $r \to \infty$), we find for the time of fall from r_1 to r

$$\Delta T = \frac{2}{3}\frac{r_g}{c}\left[\left(\frac{r_1}{r_g}\right)^{3/2} - \left(\frac{r}{r_g}\right)^{3/2}\right]. \tag{2.3.9}$$

We thus conclude that while the duration Δt, of falling for the distant observer is infinite, the time ΔT measured by the clock of the particle itself is finite. This result, which at first glance seems quite unexpected, can be given the following physical interpretation. The clock on the particle falling toward r_g is slowed down relative to the clock at infinity; first, because time is slowed down in the gravitational field [see (2.2.5)], and second, because of the Lorentz contraction of time when the velocity of the clock $v \to c$ as $r \to r_g$. As a result, the interval in t is infinite, but it becomes finite when clocked in T.

2.4. Spacetime Within the Schwarzschild Sphere

The fact that the proper time of fall to the Schwarzschild sphere is finite, suggests a method of constructing a reference frame which can be extended to $r < r_g$. The reference frame must be fixed to the falling particles. No infinite accelerations F and no corresponding infinite forces will arise at the gravitational radius of this system, because the particles of the system fall freely, are weightless, and F is identically zero everywhere. The simplest such frame of reference consists of freely falling particles that have zero velocity at spatial infinity [Lemaitre's reference frame (1933); see also Rylov (1961)]. The motion of these particles is described by equation (2.3.9).

In order to introduce this reference frame, we choose for the time coordinate the time T measured by a clock fixed to the falling particles. At a certain instant $T =$ const that we take for $T = 0$, the ensemble of freely falling particles are at different r_1. We can choose these values of r_1, which mark the particles and remain unchanged for each of them, as the new radial coordinate in this reference frame.

The squared interval in the frame of freely falling particles is written in the form

$$ds^2 = -c^2 dT^2 + \frac{r_1\, dr_1^2}{r_g \left[\left(\frac{r_1}{r_g} \right)^{3/2} - \frac{3}{2} \frac{cT}{r_g} \right]^{2/3}} + \left[\left(\frac{r_1}{r_g} \right)^{3/2} - \frac{3}{2} \frac{cT}{r_g} \right]^{4/3} r_g^2 (d\theta^2 + \sin^2\theta\, d\phi^2). \tag{2.4.1}$$

It is convenient to use instead of r_1, the following radial coordinate in the frame considered

$$R = \frac{2}{3} r_g \left(\frac{r_1}{r_g} \right)^{3/2}. \tag{2.4.2}$$

The squared interval (2.4.1) is now transformed to the form

$$ds^2 = -c^2 dT^2 + \frac{dR^2}{\left[\frac{3}{2} \frac{(R - cT)}{r_g} \right]^{2/3}} + \left[\frac{3}{2} \frac{(R - cT)}{r_g} \right]^{4/3} r_g (d\theta^2 + \sin^2\theta\, d\phi^2). \tag{2.4.3}$$

The reference frame with interval (2.4.3) (the *Lemaitre reference frame*) indeed has no singularity on the Schwarzschild sphere. In order to show that this is the case, we write the explicit relation between the Schwarzschild and Lemaitre coordinates

$$r = \left[\frac{3}{2}\frac{(R - cT)}{r_g}\right]^{2/3} r_g , \tag{2.4.4}$$

$$t = \frac{r_g}{c}\left\{-\frac{2}{3}\left(\frac{r}{r_g}\right)^{3/2} - 2\left(\frac{r}{r_g}\right)^{1/2} + \ln\left[\frac{(r/r_g)^{1/2} + 1}{(r/r_g)^{1/2} - 1}\right] + \frac{R}{r_g}\right\}. \tag{2.4.5}$$

Setting $r = r_g$ in (2.4.4), we obtain the equation for position of the Schwarzschild sphere in the Lemaitre reference frame:

$$r_g = \frac{3}{2}(R - cT) . \tag{2.4.6}$$

The expressions for all $g_{\alpha\beta}$ in (2.4.3) on the Schwarzschild sphere are quite regular, without any singularity. The calculation of all nonzero invariants of the curvature of four-dimensional spacetime also reveals no singularities on the Schwarzschild sphere. The Lemaitre reference frame extends to $r < r_g$. The spacetime in the Lemaitre R and T coordinates is shown in Figure 1 (by virtue of symmetry, the angular coordinates θ and ϕ are irrelevant). The reference frame can be extended up to $r = 0$, that is [see (2.4.4)], up to $R = cT$. Here we find the true singularity of spacetime, namely infinite curvature. We find this, for example, in the fact that the curvature invariant $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ tends to infinity as $R - cT \to 0$ The infinity in this invariant signifies the infinity of gravitational tide forces.

As shown in Figure 1, each freely falling particle with R = const in the Lemaitre reference frame moves in time T to smaller r. The particle reaches r_g over a time T,

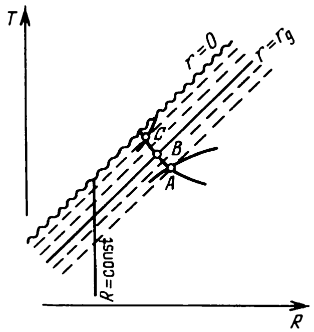


Fig. 1. Schwarzschild spacetime in Lemaitre coordinates. Dashed lines – lines of r = const; ABC – the world line of a photon falling into a black hole. Segments of the world lines of photons moving in the opposite direction are shown at points A, B, C.

keeps falling, and reaches the true singularity $r = 0$.[5] Spacetime cannot be extended beyond the singularity at which gravitational tide forces grow infinitely and particles would be destroyed. In the neighborhood of $r = 0$, quantum effect of the gravitational field become essential; this aspect is discussed in Chapter 12.

Figure 1 also shows the world lines of radial light rays. They are found from (2.4.3) by imposing the conditions $ds = 0$, $d\theta = d\phi = 0$:

$$c\frac{dT}{dR} = \pm\left[\frac{r_g}{\frac{3}{2}(R - cT)}\right]^{1/3} . \tag{2.4.7}$$

The position of light cones in Figure 1 immediately indicates why the Schwarzschild sphere plays a special role in the spherical gravitational field in general and in the Schwarzschild reference frame in particular. Indeed, if $r > r_g$, then the $r = \text{const}$ world lines (here and below we assume $\theta = \text{const}$, $\phi = \text{const}$) lie within the light cone, that is, they are time-like; the $r = r_g$ line coincides with the photon's world line, that is, it is light-like; and finally, if $r < r_g$, the $r = \text{const}$ world lines are space-like.

This is why the Schwarzschild reference frame formed by particles with $r = \text{const}$ cannot be extended to $r < r_g$.

This situation is found to be typical for general relativity and constitutes the difference between it and the ordinary field theory in flat space. Special coordinates must be chosen for solving Einstein's equations. Correspondingly, additional conditions are to be introduced in order to fix the form of metrics. In general, it is impossible to guarantee that the chosen coordinates cover the entire spacetime, because the spacetime of general relativity may have a complicated global structure (e.g., have a nontrivial topology). That was the situation encountered above in the attempts of describing the entire spherically symmetric spacetime in curvature coordinates (2.1.2). The general method of establishing whether the obtained solution indeed describes the entire spacetime or only its part is to analyze the behavior of test particles and light rays. If some of the particles reach the 'boundary' of the chosen coordinate system in a finite proper time (or for the finite value of the affine parameter for photons), and there are no physical singularities at the 'final' points of particles trajectories, then this coordinate system is incomplete. By changing the coordinate constrains and switching to metric (2.4.1), we were able to cover a greater part of the spacetime and, among other things, describe the possible events below gravitational radius. A discussion of whether the Lemaitre coordinate system is indeed complete and whether metric (2.4.1) describes the entire spacetime will be delayed until Section 2.7, but now we return to considering the properties of the Schwarzschild sphere and the region of spacetime within it. [For a general discussion of the relevant aspects, see Hawking and Ellis (1973).]

The most striking feature of the Schwarzschild sphere lies in the following. Outward-going light ray (rightward ray in Figure 1) from a point with $r > r_g$ travels to greater r and escapes to spatial infinity. For points with $r < r_g$, both rays (left- and right-ward ones in Figure 1), travel toward smaller r; they do not escape to the

spatial infinity but are 'stopped' at the singularity $r = 0$. The world line of any particle necessarily lies within the light cone. For this reason, if $r < r_g$, all particles have to move toward $r = 0$: this is the direction into the future. The motion toward greater r is possible in the region of $r < r_g$ [see Finkelstein (1958)]. It should be emphasized that this is true not only for freely falling particles (i.e. particles moving along geodesics) but also for particles moving at arbitrary acceleration. Neither radiation nor particles can escape from within the Schwarzschild sphere to the distant observer.

In (2.2.1), we defined r in such a way that $g_{22} = r^2$, that is, as the radial coordinate in curvature coordinates system [see (2.2.1)]. Formally, r within the Schwarzschild sphere is defined in the same manner, although here the r = const line is space-like and cannot serve as the radial spatial coordinate. If $r < r_g$, the quantity g_{22} is always a function of time, and a monotone function in any reference frame determined by relations (2.1.3)–(2.1.5). At $r < r_g$ all reference frames are nonstatic and both radial rays travel only to smaller r (and hence, to smaller g_{22}). Spacetime regions possessing this property are referred to as *T regions* [Novikov (1962, 1964a)]. The spacetime region outside the Schwarzschild sphere is said to form the *R region.*

Let us give a more exact definition of R and T regions. Consider a spherically symmetric spacetime. It may contain matter ($T_{\alpha\beta} \neq 0$) or it may be empty. By the definition of the spherically symmetric gravitational field, its metric can always be written in the form (2.1.2). If the x^1= const, θ = const, ϕ = const world line in the neighborhood of a given point is time-like, this point belongs to the R region. If this line is space-like, the point in question belongs to the T region.

Let us return to the case of spherically symmetric gravitational field in vacuum.

Apart from the already described Lemaitre reference frame, other reference frames are employed for analyzing a region both inside and outside the Schwarzschild sphere. Here and in the sections that follow, we describe some of these frames.

First of all, we again turn to coordinate system (2.2.1). As we have shown in Section 2.2, this system has a singularity on the Schwarzschild sphere. But if r is strictly less than r_g, the metric coefficients are again regular. Is there a straightforward physical interpretation of this system if $r < r_g$? Indeed, there is [Novikov (1961)]. As was demonstrated above, now the coordinate r ($r < r_g$) cannot be radial spatial coordinate. It can play, however, the role of the temporal coordinate, as follows from (2.2.1) where the coefficient with dr^2 reverses its sign on crossing the Schwarzschild sphere and is negative where $r < r_g$. On the other hand, now the coordinate t can be used as the spatial radial coordinate, the coefficients with dt^2 being positive for $r < r_g$. The coordinates r and t thus changed their roles when r became less than r_g. We change the variables, $r = -c\tilde{T}$, $t = \tilde{R}/c$, and rewrite (2.2.1) in the form

$$\mathrm{d}s^2 = -\left[\frac{r_g}{(-c\tilde{T})} - 1\right]^{-1} c^2\,\mathrm{d}\tilde{T}^2 + \left[\frac{r_g}{(-c\tilde{T})} - 1\right]\mathrm{d}\tilde{R}^2 +$$

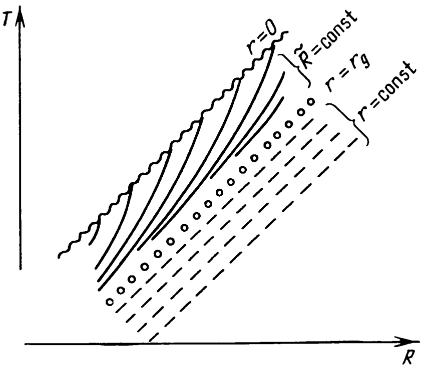


Fig. 2. $\tilde{R}$ = const world lines of particles, forming the frame of reference (2.4.9) in Lemaitre coordinates.

$$+c^2\tilde{T}^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2), \tag{2.4.8}$$

$$0<-c\tilde{T}<r_g, \quad -\infty<\tilde{R}<\infty. \tag{2.4.9}$$

The frame of reference (2.4.8)–(2.4.9) can be realized by free test particles moving along geodesic inside the sphere $r = r_g$. A three-dimensional section $\tilde{T}$= const has an infinite spatial extension along the coordinate $\tilde{R}$, while along the coordinates θ and ϕ it is closed, constituting on the whole a topological product of the sphere by a straight line. The three-dimensional volume of this section is infinite. The system is nonstationary, it contracts along θ and ϕ (the radius of the sphere decreases from r_g to 0) and expands along $\tilde{R}$. Its proper lifetime is finite:

$$\tau=\int_{r_g/c}^{0}\sqrt{-g_{00}}\ \mathrm{d}\tilde{T}=\frac{\pi}{2}r_g\,. \tag{2.4.10}$$

The world lines of the particles with $\tilde{R}$ = const that form the frame are plotted in the Lemaitre coordinates (2.4.3) in Figure 2. The figure shows that the particles move within the Schwarzschild sphere and the system is by no means an extension of the Schwarzschild system to $r < r_g$ (its world lines r = const are shown in the same figure). Time and the spatial radial direction undergo a peculiar change of roles in these systems.

Now we shall describe a reference frame which historically was the first constructed system of coordinates without singularities on r_g [Eddington (1924), Finkelstein (1958)]. This reference frame is fixed to photons which move freely along the radius. The equation of motion of photons is given by (2.3.3). The photons moving towards the center are characterized by r decreasing with t. Expression (2.3.3) for such photons can be rewritten in the form

$$t = -\frac{r}{c} - \frac{r_g}{c} \ln \left[\frac{r}{r_g} - 1 \right] + \frac{V}{c}, \tag{2.4.11}$$

where V is a constant characterizing the radial coordinate of the photon at a fixed instant t.

The logarithm in (2.4.11) is taken of the module of the difference $r/r_g - 1$, so that (2.4.11) remains valid for $r < r_g$.[6] If we take a set of photons at a fixed t and assign each photon a number V which remains unchanged in the motion of the photon, this V can be chosen as another new coordinate. This is similar to our choice of r_1 for a new radial coordinate in the case of nonrelativistic particle [see (2.3.9)]. In fact, an essential difference must not be overlooked. Namely, no observer can move together with a photon, so that in this sense the new reference frame does not fall, strictly speaking, under the definition of a reference frame. Nevertheless, such 'system' of test photons proves to be convenient. One needs to remember, though, that V is a light coordinate (neither spatial not temporal one). For a second coordinate, we can choose the familiar coordinate r. Differentiating now (2.4.11) and substituting the obtained expression for dt into (2.2.1), we find

$$ds^2 = -\left(1 - \frac{r_g}{r} \right) dV^2 + 2\, dV\, dr + r^2 (d\theta^2 + \sin^2 \theta\, d\phi^2). \tag{2.4.12}$$

Expression (2.4.12) is regular on $r = r_g$. Indeed, the coefficient with dV^2 vanishes on r_g, but the presence of the term $2\, dV\, dr$ ensures that this coordinate system remains nondegenerate. The spacetime in the coordinates V, r is shown in Figure 3. We took into account that the system is not orthogonal and the coordinate lines are at a constant angle of 45° (on the V, r plane).

2.5. Contracting and Expanding T Regions

The properties, discussed above, of reference frames within the Schwarzschild sphere in the T region are quite peculiar. Indeed, we notice that all these coordinate systems must contract along the θ and ϕ directions, and the coefficient g_{22} must decrease in time (this is equivalent to r decreasing with time). This fact can be rephrased as the inevitable motion of all light rays and all particles in the T region toward the singularity. We know that Einstein's equations are invariant under time reversal. All the formulas given above remain the solution of Einstein's equations if the following change of variables is made: $t \to -t$, $T \to -T$, $\tilde{T} \to -\tilde{T}$, $V \to -U$, where U enumerates the outgoing rays ($U = 2t - V$). However, this change is equivalent to time reversal. Hence, reference frames are possible (e.g., the Lemaitre, Eddington, etc. reference frames) which expand from below the Schwarzschild sphere and are formed by particles escaping from the singularity in the T region, intersecting later the Schwarzschild sphere, and escaping to infinity (Figures 4 and 5).

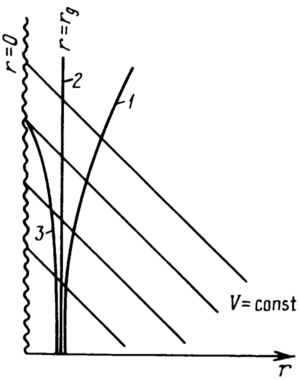


Fig. 3. Schwarzschild spacetime in Eddington–Finkelstein coordinates (2.4.12): V = const are the world lines of photons falling toward $r = 0$; 1, 2, 3 are the world lines of photons moving in the direction opposite to that of V = const.

Is this conclusion of the escape of particles from below the Schwarzschild sphere compatible with the statement, emphasized on several occasions above, that no particle can escape from this sphere? The situation is as follows. No particle can escape from a T region (or from within the Schwarzschild sphere) if this (or some other) particle *entered* it before. In other words, if the Schwarzschild sphere *can be entered*, it *cannot be escaped.* The T region which appears in solution with reversed time is a *quite different* T region, with very different properties. While only contraction was possible in the former T region, only expansion is possible in the latter one, so that nothing can fall into it (this is very clear in Figures 4 and 5).

Note that the external space (beyond the $r = r_g$ sphere) is essentially the same in two cases. Its metric is reduced to (2.2.1) by a transformation of coordinates, but it can be extended into the Schwarzschild sphere *in two ways*: either as a contracting T

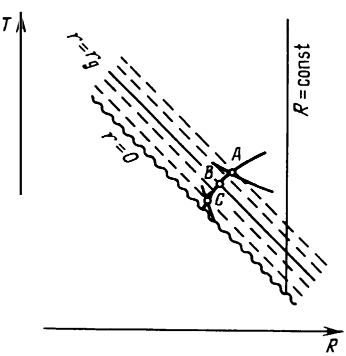


Fig. 4. Schwarzschild spacetime in expanding Lemaitre coordinates. The time arrow is reversed in comparison with Figure 2.

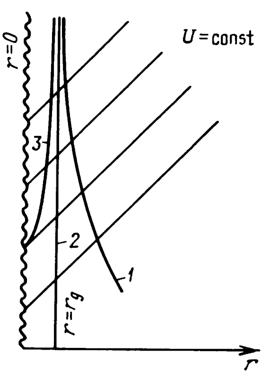


Fig. 5. Schwarzschild spacetime in expanding Eddington–Finkelstein coordinates. The time arrow is reversed in comparison with Figure 3.

region, or as an expanding T region (but these modes are incompatible!). It depends on boundary or initial conditions which type of T region is realized in a specific situation. This aspect is treated in detail in the next section. The contracting T region is usually denoted by T_-, and the expanding one, by T_+.

2.6. Formation of a Black Hole in a Gravitational Collapse. White Holes

In this section we analyze the process of formation of a black hole as a result of the contraction of a spherical mass to a size less than r_g. In order to eliminate the effects which are not directly relevant to the formation of a black hole and can only make the solution more difficult to obtain, we consider the contraction of a spherical cloud of matter at zero pressure $p = 0$ (dust cloud). There is no need then to include in the analysis the hydrodynamic phenomena due to the pressure gradient. All dust particles move along the geodesic, being subjected only to the gravitational field. The solution of Einstein's equations for this case was obtained by Tolman (1934). In the solution below, the reference frame is comoving with the matter, that is, dust particles have constant R, θ, ϕ:

$$ds^2 = -c^2\,dT^2 + g_{11}(T,R)\,dR^2 + r^2(T,R)(d\theta^2 + \sin^2\theta\,d\phi^2), \tag{2.6.1}$$

$$\dot{r}^2 = f(R) + \frac{F(R)}{r}, \tag{2.6.2}$$

$$g_{11}(T,R) = \frac{(r')^2}{1 + f(R)}, \tag{2.6.3}$$

$$\frac{8\pi G\rho}{c^2} = \frac{F'(R)}{r'r^2}. \tag{2.6.4}$$

Here a dot denotes differentiation with respect to cT, and a prime, differentiation with respect to R; $f(R)$ and $F(R)$ are two arbitrary functions of R (subject to the condition $1 + f(R) > 0$). Equation (2.6.2) defines the function r after $f(R)$ and $F(R)$ have been fixed, and ρ is the density of matter. Tolman's solution can describe, for example, the compression of a spherical dust cloud of finite size. For this description, we choose an initial moment T = const. Then (2.6.4) gives density distribution. If the coordinates R_1 determines the boundary of the sphere, then beyond the sphere (at $R > R_1$) we find $\rho = 0$ and F = const. The evolution of r with T for the particles of the sphere is described by equation (2.6.2). The equation shows that each particle with fixed R and $\dot{r} < 0$ reaches the point $r = 0$, where spacetime has a true singularity, in a finite time T.

Outside the sphere, the metric of spacetime is determined in a unique manner by the value of F at the boundary R_1. In vacuum this metric is the Schwarzschild metric (see Section 2.2).

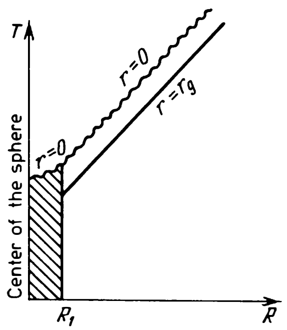


Fig. 6. Spacetime of a contracting spherical cloud creating a black hole: Lemaitre coordinates. The region inside the sphere is hatched.

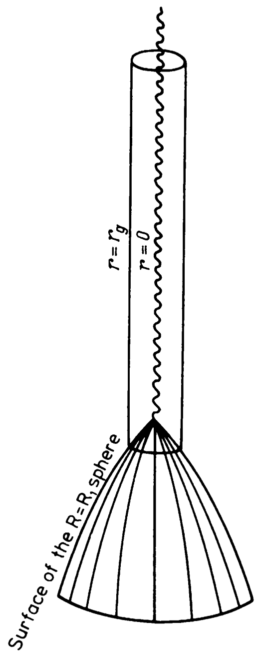


Fig. 7. Spacetime of a contracting spherical cloud creating a black hole: Eddington–Finkelstein coordinates.

The particles on the surface of the sphere are freely falling in this outer metric, so that their motion can be described as the motion along radial geodesic in the Schwarzschild metric [see (2.3.5)]. We can thus consider the contraction of a spherical cloud on whose surface the particles fall at the parabolic (escape) velocity. The motion of such particles is given by especially simple formulas [see (2.3.9)]. The equation of the world line of these particles in the Lemaitre reference frame is R = const. In Eddington–Finkelstein reference frame the equation of the same line is given parametrically by expression (2.4.11), (2.4.4), (2.4.5), provided we set $R = R_1$ = const in the last two formulas.

Figures 6 and 7 represent the spacetime in the case of the contracting sphere in the Lemaitre and Eddington–Finkelstein coordinates, respectively. Figure 7, which also shows one of the rotational degrees of freedom, is especially illustrative. The surface of the contracting spherical cloud reaches the Schwarzschild sphere $r = r_g$ in a finite proper time and then contracts to a point at $r = 0$. This process is known as the relativistic *gravitational collapse*. As a result of the collapse, a spacetime region is formed within the Schwarzschild sphere from which no signals can escape to the spatial infinity. This region is defined as the *black hole*. The relativistic gravitational collapse of a spherical nonrotating body thus generates a spherical black hole.

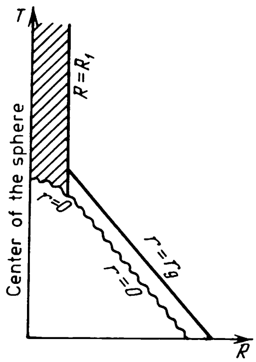


Fig. 8. Expansion of the spherical cloud from within the Schwarzschild sphere in expanding Lemaitre coordinates.

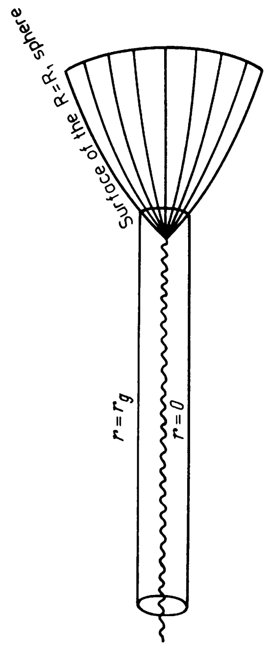


Fig. 9. Expansion of the spherical cloud from within the Schwarzschild sphere in Eddington–Finkelstein coordinates.

Note now that the assumption made above on the absence of pressure produces no qualitative changes in the picture of the birth of a spherical black body. We find the same behavior in the general case of the contraction of a sphere at a nonzero pressure ($p \neq 0$). When the surface of the contracting spherical cloud approaches the Schwarzschild sphere, no pressure can prevent the formation of the black hole [for details, see Zel'dovich and Novikov (1971)]. These aspects do not directly concern us here, so the specifics are omitted.

The gravitational collapse produces a contracting T region within the Schwarzschild sphere. This result follows from the condition of continuity of the coefficient g_{22} of the metric (the coefficient with the angular spatial term) in the transition from the contracting surface of the sphere to the vacuum at a fixed instant of time.[7] As time goes on, this coefficient decreases on the surface of the contracting sphere for $r < r_g$. Therefore, it will decrease, owing to the continuity, also outside the spherical cloud (for $r < r_g$), that is, the region within the $r = r_g$ sphere is indeed the contracting T region.

What are the conditions necessary for the formation of an expanding T_+ region? Reversing the time arrow in Figures 6 and 7, we obtain Figures 8 and 9. They represent the expansion of the spherical cloud from within the Schwarzschild sphere. Now the continuity of g_{22} at the boundary of the sphere implies that the vacuum beyond the spherical cloud but within the Schwarzschild sphere $r = r_g$ contains the expanding T_+ region. Figure 8 clearly illustrates the general situation.

Recall that the $r = 0$ line is space-like so that a reference frame exists in which all events on this line are simultaneous. Therefore, one cannot say (as one would be tempted to conclude at first glance) (see Figures 8 and 9) that *first* the singularity $r = 0$ existed in vacuum and *then* the matter of the spherical cloud began to expand from the singularity. These events are not connected by a time-like interval. It is more correct to say that the nature of the space-like singularity at $r = 0$ is such that it produces the expansion in vacuum (expanding T_+ region) to the right of R_1 and the expanding matter of spherical cloud to the left of R_1 (see Figure 8). Note that no particle coming from the spatial infinity (or from any region of $r > r_g$) can penetrate the expanding T region. Such regions of spacetime are called *white holes* [Novikov (1964b), Ne'eman (1965)]. These objects cannot appear in the Universe as a result of collapse of some body, but could be formed, in principle, in the expanding Universe at the moment the expansion set in. This range of problems is discussed in detail in Section 13.2.

To conclude the section, we again emphasize that it is mathematically impossible to extend the solution beyond the true spacetime singularity at $r = 0$. Therefore, general relativity cannot answer the question on what will happen after the contraction to $r = 0$ in a T_- region, or what was there before the start of the expansion from $r = 0$ in a T_+ region (or even say whether these questions are correctly formulated). It is physically clear that in the neighborhood of $r = 0$ quantum processes become essential for the spacetime itself (this effect is not described by general relativity); we return to this phenomenon in Chapter 13.

2.7. Eternal Black and White Holes

At first glance it seems that an eternal black hole might exist in empty space, that is, a black hole which does not appear as a result of contraction of mass (as shown in Figures 6 and 7) but exists perpetually in the form shown in Figures 1 and 3. The spacetime always contains a Schwarzschild sphere but has no contracting material cloud.

Surprisingly, it is found that the existence of such 'pure' eternal black hole is

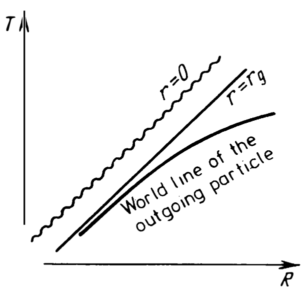


Fig. 10. World line of a particle escaping from the Schwarzschild sphere, in contracting Lemaitre coordinates.

forbidden in principle. The reason is as follows. The picture (or rather, map) of the spacetime shown in Figure 1 (or Figures 3) does not cover the entire spacetime. In order to demonstrate this, consider a particle which moves freely along the radius away from the Schwarzschild sphere. Its world line in the contracting Lemaitre coordinate system is given by the expression

$$\text{const} = -\frac{R}{r_g} + \frac{2cT}{r_g} - 4\left(\frac{3}{2}\frac{R - cT}{r_g}\right)^{1/3} + \\ + 2\ln\left\{\frac{\left[\frac{3}{2}(R - cT)\right]^{1/3} + r_g^{1/3}}{\left[\frac{3}{2}(R - cT)\right]^{1/3} - r_g^{1/3}}\right\} \tag{2.7.1}$$

and is plotted in Figure 10. Continued into the past, this line approaches asymptotically the line $r = r_g$, without intersecting it. In the time T of the Lemaitre frame the particle exists beginning from $T = -\infty$. But we know that in proper time the path from r_g to any finite r takes a finite interval of time. Therefore, Figure 10 does not cover the entire past of the particle in question from $\tau = -\infty$ in its proper clock. Indeed, the history of a lone particle (i.e., a particle that does not interact with other particles, say, a particle that is not created in an interaction) does not terminate on the Schwarzschild sphere. The world line of such a particle must either continue indefinitely in its proper time or must terminate on the true singularity of spacetime, where new physical laws take over. Consequently, the map is incomplete and does not cover the entire space.

Is it possible to construct an everywhere empty spacetime with an eternal black

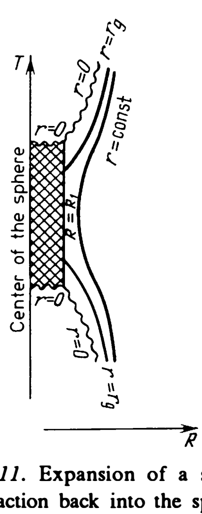


Fig. 11. Expansion of a spherical cloud from within the Schwarzschild sphere, followed by contraction back into the sphere. The region inside the spherical cloud is hatched.

hole which is complete in the sense that it covers the histories of all particles moving in this space? The answer was found to be affirmative, although it includes not only an eternal black hole but also an eternal white hole. In order to approach this construction in a natural way, consider a white hole with an expanding dust ball. Assume that the energy of motion of particles in the dust ball is such that the surface of the cloud does not escape to infinity but reaches a maximal radius and then again contracts to the size r_g and subsequently collapses to $r = 0$.

According to formula (2.3.5), the specific energy E/mc^2 of a particle on the surface of the dust cloud must be less than unity to ensure that $dr/dt = 0$ for a certain r. In Tolman's solution, (2.6.1)–(2.6.4), this expansion to a finite radius corresponds to the choice $f(R) < 0$. A qualitative representation of the spacetime with an expanding and then contracting dust ball is shown in Figure 11. This spacetime first contains a white hole and then develops a black hole. Note that the lines $r = 0$ and $r = r_g$ are represented in this solution on this figure by lines which are not straight as we had in the case of motion at a parabolic velocity (see Figures 6 and 8).

Let us begin to reduce the specific energy E/mc^2 of the particles on the surface of the dust ball, assuming the total mass M of the ball and, hence, the value of r_g to be fixed. In other words, by reducing E/mc^2 we reduce the share of the kinetic energy of outward motion in the total energy Mc^2 of the dust ball. As a result, the cloud will expand to gradually smaller radii. Finally, the cloud expands to $r = r_g$ when $E/mc^2 = 0$. A qualitative map of the spacetime is then seen in Figure 12.

What if the constant E is reduced further and made negative? At first glance, this is physically meaningless; formally, it leads to enhanced maximum expansion radius r which is found be equating dr/dt in (2.3.5) to zero. Actually, there is nothing meaningless in this operation. In order to clarify the situation, let us look at formulas (2.6.1)–(2.6.4). Assume that the dust cloud whose evolution we analyze is homogeneous. Then the spacetime metric within the cloud corresponds to the metric of the homogeneous isotropic Universe. In solution (2.6.1)–(2.6.4), this metric corresponds to choosing the functions

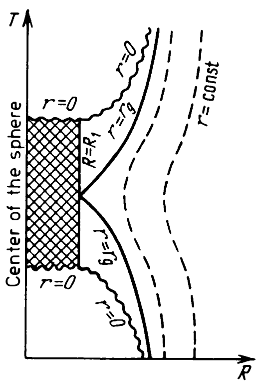


Fig. 12. The boundary of the spherical cloud expands only up to the Schwarzschild sphere and then contracts.

$$f(R) = -\sin^2 R, \tag{2.7.2}$$

$$F(R) = \alpha \sin^3 R, \tag{2.7.3}$$

where a is the scale factor determined by the density ρ_{max} within the spherical cloud at the moment of its maximum expansion:

$$a = \sqrt{\frac{3c^2}{8\pi G\rho_{max}}} . \tag{2.7.4}$$

The matter of the cloud stretches up to the boundary value of the coordinate $R = R_1$. The value R_1 may be in the range $0 < R_1 \leq \pi$. In the vacuum outside the cloud (at $R > R_1$), the particles that constitute the reference frame move freely along radial geodesics. The metric is determined by the following functions [Novikov (1963, 1964a)]:

$$f(R) = -\frac{1}{(R + \text{cotan}\, R_1 - R_1)^2 + 1} , \tag{2.7.5}$$

$$F(R) = r_g . \tag{2.7.6}$$

In this situation,

$$r_g = a \sin^3 R_1. \tag{2.7.7}$$

The sum of the masses of the particles making up the cloud equals the product of density by the sphere volume:

$$M_* = \frac{3}{4}\frac{ac^2}{G}\left(R_1 - \frac{1}{2}\sin 2R_1\right) . \tag{2.7.8}$$

The quantity M (the gravitational mass) characterizes the total energy of the particles in the cloud, the gravitational energy included. If the boundary coordinate R_1 lies in the range $\pi/2 < R_1 < \pi$, the inner region of the sphere is the so-called *semiclosed world* [see Zel'dovich and Novikov (1975) and the bibliography of earlier work therein]. In these conditions an increase in R_1 (addition of new layers of matter) increases M_* but diminishes M (because of a large gravitational defect of mass).

Our objective is to analyze the evolution of the spherical cloud when we supply progressively smaller and smaller specific energy to its particles. This means that we will take progressively smaller ratio M/M_*. In order to find the result of this change, we can take different values of the ratio M/M_* while fixing either M or M_*. This choice is of methodological significance only. When we are interested in the metric outside the dust cloud, we fix M which determines the outer metric.

Let us follow the evolution in the time of cloud boundary $r(R_1, T)$ for each fixed R_1 and find the metric outside the cloud. The ratio M/M_* is determined by (2.7.7) and (2.7.8):

$$\frac{M}{M_*} = \frac{2}{3} \sin^3 R_1 \, (R_1 - \sin R_1 \cos R_1)^{-1} \, . \qquad (2.7.9)$$

The evolution of the boundary of the cloud is described by the ratio of the maximum radius of expansion, r_{max}, to the gravitational radius r_g:

$$r_{max} / r_g = \sin^{-2} R_1 \, . \qquad (2.7.10)$$

When $R_1 \ll \pi/2$, the ratio M/M_* is only slightly less than unity; a qualitative picture of the evolution is shown in Figure 11 ($r_{max}/r_g \gg 1$). If $R_1 = \pi/2$, then

$$M/M_* = \frac{4}{3\pi} \quad \text{and} \quad r_{max}/r_g = 1;$$

the corresponding situation is shown in Figure 12. If $R_1 > \pi/2$, the dust ball is a semiclosed world and the ratio M/M_* decreases as R_1 approaches π. This corresponds to $E < 0$ in (2.3.5). Now the metric looks as shown in Figure 13.

A qualitatively new feature has emerged. The ratio r_{max}/r_g is again greater then unity. But now the boundary of the cloud does not break out from under the sphere of radius r_g in the space of the distant observer R'. A new region R'' has appeared, perfectly identical to R' outside the spherical cloud.

As $R_1 \to \pi$, the boundary R_1 gradually shifts leftward in Figure 13, leaving progressively greater fraction of R'' free. The ratio r_{max}/r_g tends to infinity, and the ration M/M_*, to zero.

In the limit $R_1 = \pi$, the region occupied by matter vanishes, leaving the entire spacetime empty (Figure 14). It contains a white hole T_+, a black hole T_-, and two identical outer spaces $R'$and R'' which transform into Euclidean spaces at spatial

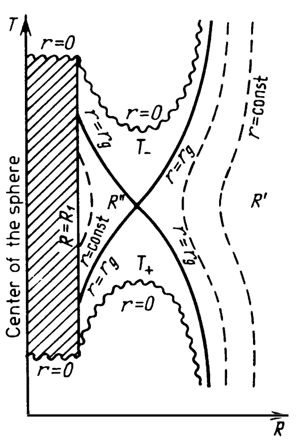


Fig. 13. Expansion and contraction of a semiclosed world.

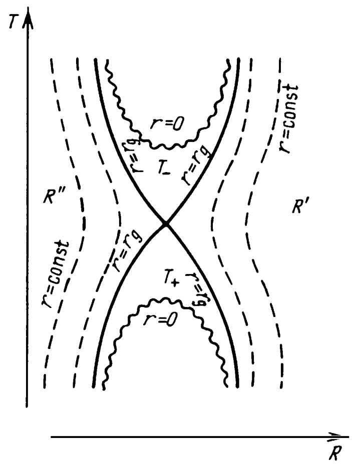


Fig. 14. Everywhere empty spacetime with a white and a black hole.

infinity. This spacetime is complete in the sense that any geodesic now either continues indefinitely or terminates at the true singularity.

The reference frame covering the entire spacetime in Figure 14 is described by a solution of type (2.7.5)–(2.7.6), where now it is convenient to place the origin of R at the minimum of the function $f(R)$. Then

$$f(R) = -\frac{1}{R^2+1}, \quad F = r_g\,, -\infty < R < \infty, \tag{2.7.11}$$

The complete, everywhere empty spacetime shown in Figure 14 was first constructed by Synge (1950) and then by Fronsdal (1959), Kruskal (1960), Szekeres (1960). The physical arguments given above and the solution (2.7.11) were obtained by Novikov (1962b, 1963).

We have thus obtained an everywhere empty space with a white and black holes (essentially together). These holes can be described as 'eternal' because for distant observers which are at rest in R' and R'' these holes are eternal.

The physical meaning of the second 'outer space' R'' has become clear above, where we described the evolution of a spherical cloud with progressively lower specific energy M/M_*. In Section 13.2 we discuss whether eternal black and white holes (similar to those in Figure 14) completely devoid of matter can really exist; this is related to the problem of stability of white holes.

To conclude this section, we give the coordinate system suggested by Kruskal (1960) and Szekeres (1960). As system (2.7.11), this one covers the entire spacetime of eternal white and black holes. In these coordinates, the interval is written in the form

$$ds^2 = \frac{4r_g^3}{r}\, e^{-r/r_g}(-d\tilde{v}^2 + d\tilde{u}^2) + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{2.7.12}$$

where r is a function of $\tilde{v}$ and $\tilde{u}$:

$$\left(\frac{r}{r_g} - 1\right) e^{r/r_g} = \tilde{u}^2 - \tilde{v}^2. \tag{2.7.13}$$

The following formulas give the relation of the coordinates $\tilde{v}$ and $\tilde{u}$ with r and t in the regions R' and T_-:

$$\left.\begin{aligned} \tilde{u} &= \left(\frac{r}{r_g} - 1\right)^{1/2} e^{r/2r_g} \cosh\frac{ct}{2r_g} \\ \tilde{v} &= \left(\frac{r}{r_g} - 1\right)^{1/2} e^{r/2r_g} \sinh\frac{ct}{2r_g} \end{aligned}\right\} \text{ for } r > r_g\,, \tag{2.7.14}$$

$$\left.\begin{aligned}\tilde{u} &= \left(1 - \frac{r}{r_g}\right)^{1/2} e^{r/2r_g} \sinh \frac{ct}{2r_g} \\ \tilde{v} &= \left(1 - \frac{r}{r_g}\right)^{1/2} e^{r/2r_g} \cosh \frac{ct}{2r_g}\end{aligned}\right\} \text{ for } r < r_g. \qquad (2.7.15)$$

Similar relations in the regions R'' and T_+ are obtained by the change of variables $\tilde{u} \to -\tilde{u}$, $\tilde{v} \to -\tilde{v}$. The Kruskal coordinate system's convenience lies in that the radial null geodesics are always plotted by straight lines inclined at an angle of 45° to coordinate axes.

2.8. Celestial Mechanics in the Gravitational Field of the Black Hole

Let us return to processes in the space which is exterior with respect to the Schwarzschild sphere of the black hole. In this section, we consider the motion of test particles along geodesics in the gravitational field of a black hole. These phenomena were analyzed in detail a long time ago and included in text books and monographs [see e.g. Zel'dovich and Novikov (1971), Misner, Thorne, and Wheeler (1973)]. Here we will briefly discuss those features of motion which are specific for black holes, not just for the strong gravitational field (say, the field around the neutron star).

We consider the motion of particles in the space exterior with respect to the Schwarzschild reference frame, clocked by an observer at infinity (Section 2.2). The gravitational field being spherically symmetric, the trajectory of a particle is planar; we can assume it to lie in the plane $\theta = \pi/2$. The equations of motion have the form

$$\left(\frac{\mathrm{d}r}{c\,\mathrm{d}t}\right)^2 = \frac{(1 - r_g/r)^2\,[\tilde{E}^2 - (1 - r_g/r)(1 + \tilde{L}^2 r_g^2/r^2)]}{\tilde{E}^2}, \qquad (2.8.1)$$

$$\frac{\mathrm{d}\phi}{c\,\mathrm{d}t} = \frac{(1 - r_g/r)\tilde{L}r_g}{\tilde{E}r^2}. \qquad (2.8.2)$$

Here $\tilde{E}$ is the specific energy of a particle (per unit proper energy mc^2, m being the mass of the particle), and $\tilde{L}$, is the specific angular momentum (measured in units of mcr_g). These two quantities are conserved in the course of motion. The physical velocity v of the particle, measured by a nearby observer by his clock, τ, is directly related to the energy $\tilde{E}$, as we find from (2.8.1)–(2.8.2):

$$\tilde{E}^2 = (1 - r_g/r)(1 - v^2/c^2)^{-1}. \qquad (2.8.3)$$

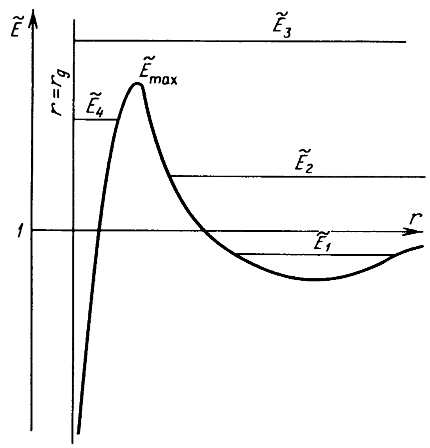


Fig. 15. Effective black hole potential.

The qualitative features of the motion are revealed in the following way. Setting dr/dt equal to zero, we find the points of maximum approach of the particle to the black hole and the maximum distance from it. The right-hand side of (2.8.1) vanishes when the equality

$$\tilde{E}^2 = (1 - r_g/r)(1 + \tilde{L}^2 r_g^2/r^2) \tag{2.8.4}$$

is satisfied. This expression is sometimes called the *effective potential.* A typical curve (2.8.4) for a fixed $\tilde{L}$ is plotted in Figure 15.

The specific energy of moving particle remains constant; in Figure 15 this motion is shown by a horizontal line. Since the numerator of (2.8.1) must be positive, the horizontal segment representing the motion of the particle lies above the (2.8.4) curve. The intersection of the horizontal line with the effective potential gives the points of the maximum approach to, and the maximum separation from, the black hole. The trajectory of the motion of a particle is not a conic section, and, in general, is not closed. Figure 15 demonstrates horizontals for typical motions. The horizontal $\tilde{E}_1 < 1$ corresponds to the motion in a bounded region in space between r_1 and r_2; this is an analogue of the elliptic motion in Newtonian theory (an example of such a trajectory is shown in Figure 16a).[8] The segment $\tilde{E}_2 > 1$ corresponds to a particle arriving from infinity and again moving away to infinity (an analogue of the hyperbolic motion). An example of this trajectory is plotted in Figure 16b.

Finally, the segment $\tilde{E}_3$ does not intersect the potential curve but passes above its maximum $\tilde{E}_{max}$; it corresponds to a particle falling into the black hole (gravitational capture). This type of motion is impossible in Newtonian theory and is typical for the black hole. The trajectory of this motion is shown in Figure 16c. The gravitational capture (see the next section) becomes possible owing to the maximum in the effective potential. No such maximum appears on the corresponding curve of Newtonian theory.

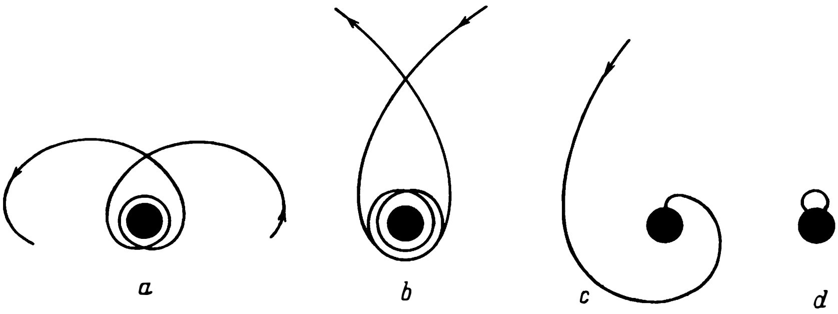


Fig. 16. Trajectories of particles with energies (a) $\tilde{E}_1$, (b) $\tilde{E}_2$, (c) $\tilde{E}_3$, and (d) $\tilde{E}_4$.

In addition, another type of motion is possible in the neighborhood of a black hole, namely, that corresponding to the horizontal segment $\tilde{E}_4$ in Figure 15. This line may lie below or above unity (in the latter case, for $\tilde{E}_{max} > 1$), stretching from r_g to the intersection with the curve $\tilde{E}(r)$. This segment represents the motion of a particle which, for example, first recedes from the black hole and reaches r_{max} [at the point of intersection of $\tilde{E}_4$ and $\tilde{E}(r)$] but then again falls toward the black hole and is absorbed by it (Figure 16d).

A body can escape to infinity if its energy $\tilde{E} \geq 1$. From equation (2.8.3) we find that the expression for the escape velocity (v_{esc} corresponds to $\tilde{E} = 1$) is

$$v_{esc} = c\sqrt{r_g/r} = \sqrt{2GM/r} \ , \tag{2.8.5}$$

which coincides with the expression given by Newtonian theory.

Note that in Newtonian theory, the escape velocity in the field of a point-like mass guarantees the escape to infinity regardless of the direction of motion. The case of the black hole is different. Here trajectories are possible that terminate in the black hole (of type $\tilde{E}_4$ or $\tilde{E}_3$ in Figure 15, the latter occurring if the particle moves towards the black hole). We have already called this effect the *gravitational capture*. Circular motion around a black hole is an important particular case of motion of a particle, in which $dr/dt \equiv 0$. This motion is shown in Figure 15 by a point at the extremum of the effective potential curve. A point at the minimum corresponds to a stable motion, and that at the maximum, to an unstable motion. The latter mode has no analogue in Newtonian theory and is specific of black holes. Of course, no real motion of a point with $\tilde{E}$ equal to $\tilde{E}_{max}$ for a given $\tilde{L}$, that is, motion along an unstable curvilinear orbit, is possible, just as any motion along unstable trajectories would be impossible. If, however, the motion of a particle is represented by a horizontal line $\tilde{E}$ = const very close to $\tilde{E}_{max}$, then the particle makes many turns around the black hole at an r corresponding to $\tilde{E}_{max}$ before the orbit moves far away from this value of r . An example of this motion is shown by the orbit of Figure 16b. The shape and position of the potential $\tilde{E}(r)$ are different for different $\tilde{L}$: the corresponding curves for some values of $\tilde{L}$ are shown in Figure 17.

The maximum and minimum appear on $\tilde{E}(r)$ curves when $L > \sqrt{3}$. If $\tilde{L} < \sqrt{3}$, the

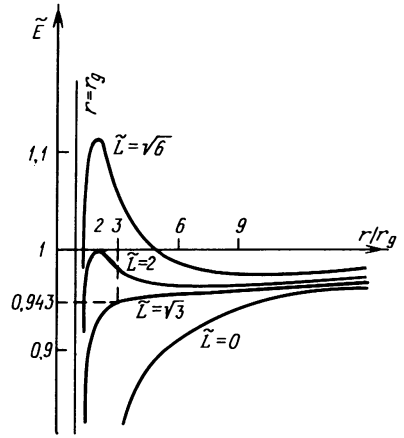


Fig. 17. Effective potentials for different values of $\tilde{L}$.

$\tilde{E}(r)$ curve is monotone. Hence, the motion on circular orbits is possible only if $\tilde{L} > \sqrt{3}$. The minima of the curves then lie at $r > 3R_g$. Stable circular orbits thus exist only for $r > 3r_g$ [Hagihara (1931)]. At smaller distance, there are only unstable circular orbits corresponding to the maximum of $\tilde{E}_{max}$ curves. If $\tilde{L} \to \infty$, the coordinates of the maximum on $\tilde{E}_{max}$ curves decrease to $r = 1.5r_g$. Even unstable circular motion becomes impossible at r less than $1.5r_g$.

The critical circular orbit that separates stable motions from unstable ones corresponds to $c = 3r_g$. Particles move along it at a velocity $v = c/2$, the energy of a particle being $\tilde{E} = \sqrt{8/9} \approx 0.943$. This is the motion with the maximum possible binding energy $E \approx 0.057mc^2$. The velocity of motion on (unstable) orbits with $r < 3r_g$ increases, as r decreases, from $c/2$ to c on the last circular orbit with $r = 1.5r_g$. When $r = 2r_g$, the particle's energy is $\tilde{E} = 1$, that is, the circular velocity is equal to the escape velocity. If r is still smaller, the escape velocity is smaller then the circular velocity. There is no paradox in it, since the circular motion here is unstable and even the tiniest perturbation (supplying momentum away from the

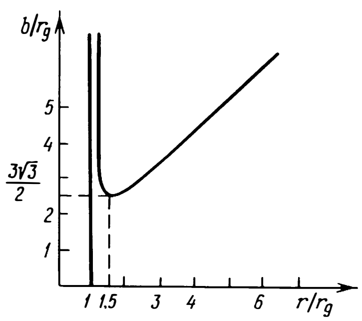


Fig. 18. The position of extremums in r on the trajectory of an ultrarelativistic particle as a function of impact parameter b.

black hole) transfers the particle to an orbit removing it to infinity, that is, an orbit corresponding to hyperbolic motion.

Let us consider the motion of an ultrarelativistic particle. In (2.8.1), (2.8.2) it corresponds to the limit $v \to c$, so that $\tilde{E} \to \infty$ and $\tilde{L} \to \infty$. One must keep in mind that the ratio $\tilde{E}/\tilde{L}$ is always equal to r_g/b, where b is the impact parameter of a particle at infinity. In view of this remark, we obtain, instead of (2.8.1), (2.8.2),

$$\left(\frac{dr}{c\,dt}\right)^2 = \left(1 - \frac{r_g}{r}\right)^2 \left[1 - \frac{b^2}{r^2}\left(1 - \frac{r_g}{r}\right)\right], \tag{2.8.6}$$

$$\frac{d\phi}{c\,dt} = \left(1 - \frac{r_g}{r}\right)\frac{b}{r^2}. \tag{2.8.7}$$

Formulas (2.8.6), (2.8.7) describe the bending of the trajectories of an ultrarelativistic particle and a light beam moving close to the black hole. Setting the expression in brackets in (2.8.6) equal to zero, we find the position of the extremum points on the trajectory as a function of radius r. The corresponding $b(r)$ curve is shown in Figure 18. The sign of b depends on the sense of motion; we assume that b is positive. In this figure the motion of an ultrarelativistic particle with a given b is shown by a horizontal line b = const. A particle approaches the black hole, passes by it at the minimal distance corresponding to the point of intersection of b = const with the right-hand branch of the $b(r)$ curve, and again recedes to infinity. If the intersection occurs close to the minimum $b_{min} = 3\sqrt{3} \times r_g/2$, the particle may go through a large number of turns before it flies away to infinity. The exact minimum of the curve $b(r)$ corresponds to (unstable) motion on a circle of radius $r = 1.5r_g$ at the velocity $v = c$. Note that the left-hand branch of $b(r)$ in Figure 18 corresponds to the maximum distance between the ultrarelativistic particle and the black hole; the particle first recedes to $r < 1.5r_g$ but then again falls on the black hole. Obviously, the parameter b does not have the literal meaning of the impact parameter at infinity since the particle never recedes to infinity. For a given coordinate r, this parameter can be found as a function of the tangent of the angle ψ between the trajectory of the particle and the direction to the center of the black hole:

$$b = \frac{r\,|\tan\psi|}{\sqrt{(1 - r_g/r)(1 + \tan^2\psi)}}. \tag{2.8.8}$$

If an ultrarelativistic particle approaches the black hole on the way from infinity and the parameter b is less than the critical value $b_{min} = 3\sqrt{3}\,r_g/2$, this particle falls into the black hole.

2.9. Gravitational Capture

In this section we deal with such a motion of test particle in which its trajectory terminates in the black hole. Two types of such motion are possible. First the trajectory of particle starts at infinity and ends in the black hole; second, the trajectory starts and ends in the black hole. Of course, a particle cannot be ejected from the black hole. Hence, the motion on second-type trajectory becomes possible either if the particle was placed on this trajectory via a nongeodesic curve or the particle was created close to the black hole. [9]

The gravitational capture of a particle coming from infinity is of special interest. Let us have a better look at this case.

It is clear from the analysis of motion given in the preceding section that a particle coming from infinity can be captured if its energy is greater, for a given $\tilde{L}$, then the maximum ($\tilde{E}_{max}$) on the curve $\tilde{E}(r)$. Let us consider the gravitational capture in two limiting cases, one for a particle whose velocity at infinity is much lower than the speed of light ($v_\infty/c \ll 1$) and another for a particle which is ultrarelativistic at infinity.

In the former case, $\tilde{E} = 1$. The curve $\tilde{E}(r)$, which has $\tilde{E}_{max} = 1$, corresponds to $\tilde{L}_{cr} = 2$ (see Figure 17). The maximum of this curve lies at $r = 2r_g$. Hence, this radius is minimal for the periastra of the orbits of the particles with $v_\infty = 0$ which approach the black hole and again recede to infinity. If $\tilde{L} \leq 2$, gravitational capture takes place. Therefore, the impact parameter corresponding to the capture is $b_{cr} = \tilde{L}_{cr}/\tilde{E} = 2r_g(v_\infty/c)$. The capture cross-section for a nonrelativistic particle is

$$\sigma_{nonrel} = \pi b_{cr}^2 = 4\pi(v_\infty/c)^2 r_g^2 . \tag{2.9.1}$$

For an ultrarelativistic particle, $b_{cr} = 3\sqrt{3}\, r_g/2$, and the capture cross-section is

$$\sigma_{rel} = \frac{27}{4}\pi r_g^2 . \tag{2.9.2}$$

Owing to a possible gravitational capture, not every particle whose velocity exceeds the escape limit flies away to infinity. In addition, it is necessary that the angle ψ between the direction to the black hole center and the trajectory be greater than a certain critical value ψ_{cr}. For the velocity equal to the escape threshold this critical angle is given by the expression

$$\tan \psi_{cr,\, esc} = \pm \frac{2\sqrt{(1 - r_g/r) r_g/r}}{\sqrt{1 - 4r_g/r(1 - r_g/r)}} . \tag{2.9.3}$$

The plus sign is chosen for $r > 2\, r_g$ ($\psi_{cr} < 90°$), and the minus sign is chosen for $r < 2\, r_g$ ($\psi_{cr} > 90°$).

For an ultrarelativistic particle, the critical angle is given by the formula

$$\tan \psi_{\mathrm{cr,\,rel}} = \pm \frac{\sqrt{1 - r_g/r}}{\sqrt{r_g/r - 1 + \frac{4}{27}(r/r_g)^2}} . \tag{2.9.4}$$

The plus sign is taken for $r > 1.5\, r_g$ and the minus for $r < 1.5\, r_g$.

2.10. The Motion of Particles Corrected for Gravitational Radiation

In relativistic theory, celestial mechanics differs from the Newtonian one in an additional factor not yet discussed above: emission of gravitational waves by accelerated bodies. As a result, the energy $\tilde{E}$ and angular momentum $\tilde{L}$ are not strictly the integrals of motion.

The emission of gravitational waves decelerates the moving particle (it loses energy and angular momentum). The force of deceleration is related to the interaction of the test particle of mass m with its own gravitational field and is proportional to m^2, while the interaction with the external field is proportional to mM. Therefore, if m/M is small, the force of 'radiative friction' is a small correction to the main force and the motion of the test particle is almost indistinguishable from the motion along a geodesic.

Nevertheless, these small corrections may accumulate over long periods of time and thus cause appreciable deviations of the motion from the initial trajectory.

Let us evaluate the change in the cross-section of the capture of a test particle approaching the black hole from infinity, taking into account the emission of gravitational waves and the process of gradual capture of the body circling the center [Zel'dovich and Novikov (1964, 1971)].

The gravitational radiation is calculated by analyzing the small perturbations of the Schwarzschild metric [Zerilli (1970a), Davis *et al.* (1971)]. The analysis shows that the evaluation of the changes in motion within the framework of the weak field theory and for nonrelativistic velocities gives a good approximation in all interesting cases.

Let us take the case of the cross-section for a particle approaching from infinity. As a result of the gravitational radiation, which mostly occurs at the periastron of the orbit, the particle cannot escape to infinity after it passes by the black hole (as would happen without radiation) but switches to a bound elongated orbit which all bring the particle black to the black hole; emission will occur again at the periastron, etc., until the particle falls into the hole. Taking this into consideration, one comes to the following approximate formula for the capture cross-section of a particle of mass m and velocity v_∞ at infinity:

$$\sigma_{\mathrm{grav.\ rad.}} \approx \pi \left(\frac{c}{v_\infty} \right)^2 (2x)^{2/7} r_g^2 , \quad x \gg 2^6 , \tag{2.10.1}$$

where $x = (c^2/v^2)\, m/M$. If $x \lesssim 2^6$, the cross-section coincides with (2.9.10) for nonrelativistic particles.

After the first passage past the black hole at a distance r_1 at the periastron, the particle recedes to a maximum distance (apoastron) given by the approximate formula

$$l_{\max} \approx \frac{r_g}{2\left[\dfrac{m}{M}\left(\dfrac{r_g}{r_1}\right)^{3/5} - \dfrac{v_\infty^2}{2c^2}\right]} . \tag{2.10.2}$$

If r_1 is small, $l_{\max}$ rapidly decreases after subsequent passages. Ultimately, the particle falls on the black hole.

Let us look now at the effect of gravitational radiation on the circular motion of particles. If a particle moves at $r \gg r_g$, the gradual decrease of the orbital radius obeys the following law [Landau and Lifshitz (1975)]:

$$\frac{dr}{dt} = -\frac{8}{5} c \left(\frac{m}{M}\right)\left(\frac{r_g}{r}\right)^3 . \tag{2.10.3}$$

This process lasts until the limiting stable circular orbit at $r = 3r_g$ is reached. At this orbit the binding energy $E \approx 0.057\, mc^2$ (see p. 31). This energy is emitted during the entire preceding motion. The energy emitted by the particle during one revolution on the critical circle $r = 3r_g$ is $\Delta E \approx 0.1\, mc^2\, (m/M)$. Then the particle slips into a spiral fall into the black hole; this takes about $(M/m)^{1/3}$ additional revolutions. The amount of energy radiated away at this stage is much less than that lost before $r = 3r_g$ was reached.

Chapter 3

Wave Fields Around a Spherical Black Hole

3.1. Weak Fields in the Schwarzschild Metric

In this chapter we describe the evolution of physical fields in the external field of a spherical black hole. Once this evolution is known, one can successfully study various processes in black hole physics. Among such processes there are, for example, the radiation of gravitational, electromagnetic, and other waves by particles falling on the black hole, nonspherical gravitational collapse giving rise to a black hole, scattering of various types of waves falling on the black hole from the outside, and some others.

The fields are assumed to be weak in the sense that their energy-momentum tensor slightly perturbs the black hole metric so that this effect on the background metric can be neglected.

Furthermore, we shall deal only with 'classical' fields with zero mass and integral spin (bibliography will be given for other fields). Of course, following the general plan of the book, this chapter considers only the classical field theory. The discussion of the quantum theory is delayed until Chapters 9 and 10.

Here we give the formulation of the problem, outline the method of the solution, and state the physical conclusions. The complete mathematical treatment is given in Chandrasekhar's book (1983) and a physically clear representation of the main aspects can be found in the monograph of Misner, Thorne, and Wheeler (1973) and in a review by Thorne (1972).

Gravitational perturbations of the Schwarzschild metric, which are a particular case of the fields to be considered here, are of special interest. They are treated in detail in Sections 3.2 and 3.3. The physics of the relevant phenomena was outlined with special clarity in Thorne's review (1976).

The reader will find an analysis of the behavior of nonclassical fields in papers by Hartle (1971, 1972), Teitelboim (1972a, b, c), Bekenstein (1972a, b), Detweiler (1980), Sibgatullin (1984), and Chandrasekhar (1979b, 1983). Sibrgatullin's book (1984) presents, in addition to the evolution of neutrino fields, a number of aspects of the mathematical theory of wave processes in the vicinity of the black hole. Special attention is paid to electrically charged black holes that we discuss later (see Chapter 8). The cited papers contain detailed bibliographies of the original publications.

Consider a weak zero-mass field with integral spin s in an external Schwarzschild metric. If $s = 2$, this is the case of the weak gravitational perturbation of the background metric.

It can be shown that a complete set of gauge-invariant dynamic variables can be found for each field, that is, a set of functions $\Phi^{(s)}$, defined in the outer spacetime of the black hole, such that

(1) $\Phi^{(s)}$ and $\partial\Phi^{(s)}/\partial t$ can be fixed arbitrarily at the initial instant;
(2) after (1) has been fixed, the evolution of $\Phi^{(s)}$ is completely determined by a single wave equation;
(3) once $\Phi^{(s)}$ are known, it is possible to calculate all the parameters of the field of interest by applying to $\Phi^{(s)}$ the differential operators and algebraic transformations;
(4) a method is known which calculates $\Phi^{(s)}$ from the given parameters of the field (regardless of the gauge).

The knowledge of $\Phi^{(s)}$ is equivalent to knowing the evolution of filed of the interest. The problem thus reduces to finding $\Phi^{(s)}$. The total solution of this problem required the efforts of a large number of physicists; see Regge and Wheeler (1957), Edelstein and Wishveshwara (1970), Zerilli (1970a, b), Price (1972a, b), Thorne (1972), Press and Bardeen (1971), Bardeen and Press (1973), Moncriff (1974a), and Chandrasekhar and Detweiler (1975a). An up-to-date presentation of the problem can be found in Chandrasekhar's monograph (1983).

The general approach to solving the problem is as follows. The field is expanded in spherical harmonics (scalar harmonic for $s = 0$, vector harmonics for $s = 1$, tensor harmonics for $s = 2$, etc.). Each spherical harmonic is characterized, among other things, by the multiple number l: $l = 0$ for a monopole, $l = 1$ for a dipole, and so on. Multipoles with $l < s$ do not evolve with time, so that we consider only nontrivial multipoles with $l \geq s$. There are the so-called radiative multipoles.

Price (1972a, b) was able to show that for each radiative multipole of any spin-s field there is a scalar field $\Phi_l^{(s)}$ that depends only on r and t, such that its differentiation and algebraic operations yield all the components of the original field of a given multipolarity. Each such scalar function $\Phi_l^{(s)}$ satisfies the equation [see also (3.2.1)]

$$\frac{\partial^2\Phi_l^{(s)}}{\partial r_*^2} - \frac{\partial^2\Phi_l^{(s)}}{c^2\,\partial t^2} = V_l^{(s)}(r)\,\Phi_l^{(s)}, \tag{3.1.1}$$

where $V_l^{(s)}(r)$ is effective potential determining the evolution of the field $\Phi^{(s)}$ and r_* is given by (3.2.1). This effective potential is a function of l and s (and, of course, of r and M as well).

For the scalar, vector, and tensor fields we find, respectively,

$$V_l^{(0)} = \left(1 - \frac{r_g}{r}\right)\left[\frac{r_g}{r^3} + \frac{l(l+1)}{r^2}\right], \tag{3.1.2a}$$

$$V_l^{(2)} = \left(1 - \frac{r_g}{r}\right)\left[\frac{l(l+1)}{r^2}\right], \tag{3.1.2b}$$

$$V_l^{(2)} = \left(1 - \frac{r_g}{r}\right)\left[-\frac{3r_g}{r^3} + \frac{l(l+1)}{r^2}\right]. \tag{3.1.2c}$$

Despite the differences in the forms of these potentials, they are not very different and their asymptotical behavior as $r \to r_g$ and $r \to \infty$, as well as other properties determining the evolution of wave fields, are quite similar. For this reason the evolution of radiative multipoles of the fields with unequal values of s is also similar. As a result, it is sufficient, in many problems, to analyze the behavior of only one of the fields.

We already mentioned that the object of special interest is the behavior of the gravitational perturbations of the Schwarzschild metric, that is, of gravitational waves. This is a particular case of an $s = 2$ field.

Let us discuss this case in more detail.

3.2. Gravitational Perturbations of the Schwarzschild Metric

We shall consider small perturbations of the Schwarzschild metric mostly following the analysis of Thorne (1976).

An arbitrary perturbation of the metric can be expanded, in accordance with the general approach outlined in Section 3.1, in tensor spherical harmonics characterized by the numbers l and m (where $|m| \leq l$) and the parity $\pi = (-1)^l$ for 'even' and $\pi = (-1)^{l+1}$ for 'odd' perturbations.

A perturbation with $l = 0$ describes the change in the black hole mass. A perturbation with $l = 1$ describes an increment of small angular momentum (rotation of the black hole). This aspect will treated latter.[1] These two types of perturbation do not evolve with time.

Let us turn to the radiative multipoles with $l \geq$ s, The analysis is facilitated by introducing a new radial coordinate [Wheeler (1955)],

$$r_* = r + r_g \ln(r/r_g - 1). \tag{3.2.1}$$

For fixed $l \geq 2$, m, π, there is only one dynamic variable Φ which depends only on r_*, t, and l (we drop the indices of Φ) and which satisfies, in the absence of field sources, the Regge–Wheeler equation (1957)

$$\frac{\partial^2\Phi}{\partial r_*^2} - \frac{\partial^2\Phi}{c^2\partial t^2} = V^{(2)}\Phi, \tag{3.2.2}$$

where $V^{(2)}$ is given by (3.1.2c). The variable r_* varies from $-\infty$ (as $r \to r_g$) to $+\infty$ (as $r \to \infty$). The potential $V^{(2)}$, written in terms of coordinates r_*, has the following properties: It is distinct from zero appreciably only if r_* is in the neighborhood of

zero ($r \approx 1.5\, r_g$); as $r_* \to \pm\infty$, $V^{(2)}$ falls off rapidly. We conclude, therefore, that $V^{(2)}(r_*)$ behaves as a potential barrier. Hence, equation (3.2.2) describes the one-dimensional transmission of a wave through the potential barrier. The properties of the solutions of this problem are well known. In particular, short waves ($\lambda \sim r_g$, λ being the wavelength) are easily transmitted through it. Waves with $\lambda \ll r_g$ are partly transmitted and partly reflected from the barrier, and finally, waves with $\lambda \gg r_g$ are completely reflected.

The time-dependence of the wave of frequency ω (in general, the frequencies ω are complex, the imaginary part describing the decay or growth of the wave amplitude) is given by the factor $e^{i\omega t}$. For each radiative multipole of $l \geq 2$ and some special values of ω, there exist solutions to equations (3.2.2) having no incident waves either from $r_* = -\infty$ or from $r_* = +\infty$. These are so-called *quasinormal modes* of a vibration of a black hole. The corresponding frequencies are known as the *eigenfrequencies* of the black hole. The existence of such solutions was indicated by Chandrasekhar and Detweiler (1975a). These authors also found eigenfrequencies for several multipoles; see Table I. (These frequencies are listed in units of $[GM/c^3]^{-1} = 2\pi(32{,}312\text{ Hz}) \times (M/M_\odot)^{-1}$.)

Table I

	$l = 2$	$l = 3$	$l = 4$
	0.37367+0.08896i	0.59945+0.09271i	0.80918+0.09416i
ω	0.34844+0.27469i	0.58201+0.28116i	0.79657+0.28439i
		0.42629+0.37273i	0.56010+0.42329i

[see also S. Iyerand C. Will (1987), S. Iyer (1987)].

All eigenfrequencies listed in the table have positive imaginary parts. This means that oscillations decay with time. If $\omega = \sigma + \alpha i$, the dependence on time is given by $e^{i\omega t} = e^{i\sigma t - \alpha t}$, so that the decay of oscillation amplitude at fixed r_*, is written in the form

$$\frac{\text{amplitude after one oscillation}}{\text{initial amplitude}} = \exp\left(\frac{-2\pi\alpha}{\sigma}\right). \tag{3.2.3}$$

(Frequency σ must not be confused with notation σ for the capture cross-section on p. 33 etc.) For the slowest-decaying mode $l = 2$, this ratio is 0.22. The waves due to decay of eigenoscillations partly escape to infinity ($r_* = +\infty$) and partly go into the black hole ($r_* = -\infty$).

It will be shown later that the eigenmodes of a black hole are intensively excited by bodies incident on it, are generated in a nonsymmetric gravitational collapse that forms an initially nonsymmetric black hole which, with time, reaches an equilibrium (after oscillations), etc.

If eigenfrequencies are such that $\alpha > 0$, oscillations are damped out; if $\alpha < 0$, they are enhanced. This second case signifies instability of eigenfrequencies in the linear approximation.

In Table I, all $\alpha > 0$. It can be shown [e.g., see Thorne (1976)] that $\alpha > 0$ for all eigenfrequencies. This does not mean, however, that the spherical black holes are

stable (in the linear approximation) because quasinormal modes do not form a complete set of dynamic variables. It is found, however, that the stability of the black hole (in the linear approximation) can be proved for each multipole mode (i.e., for fixed l, m, π) of perturbations using equations (3.2.2) [Thorne (1976)]. Furthermore, it can be shown that any small perturbation of the gravitational field around a spherical black hole decays with time [Thorne (1976), Moncriff (1974a, b), Wald (1979a, 1980), Chandrasekhar (1983)]. Gravitational waves partly transfer this perturbation to infinity and partly into the black hole. The stability of a black hole with respect to arbitrary perturbation modes means that the black hole is completely stable (of course, in the linear approximation).

The gravitational radiation produced in response to a perturbation of the field around a black hole, for example, that caused by bodies falling into it or by the formation of the black hole in weakly nonsymmetric collapse, can be divided (conditionally) into three components:

(1) radiation emitted directly by the source of perturbation;
(2) radiation due to the damped oscillations of quasinormal modes excited by the perturbation source (called 'ringing radiation');
(3) the so-called 'tails' of radiation, caused by scattering of gravitational waves by the effective potential.

A distant observer first records the radiation from the perturbation source, then the 'ringing' due to the quasinormal modes of oscillations of the black hole (these modes decay exponentially), and finally the 'tails' of radiation. The tails damp out much slower, namely, by the power law. This behavior actually signifies that the black hole tends asymptotically to equilibrium state. The details are to be discussed later (see Section 3.4; for additional details, see Leaver (1985, 1986a, b)]. Here we note that the amplitude of the 'tail' radiation is negligible in comparison with the first two components.

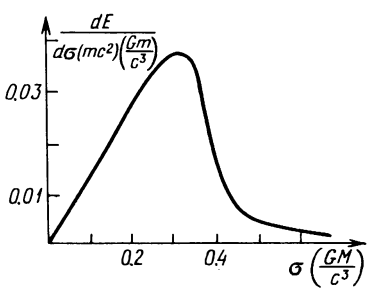


Fig. 19. Spectrum of gravitational radiation of a particle that falls radially into a black hole with $v_\infty = 0$; the spectrum is averaged over all directions.

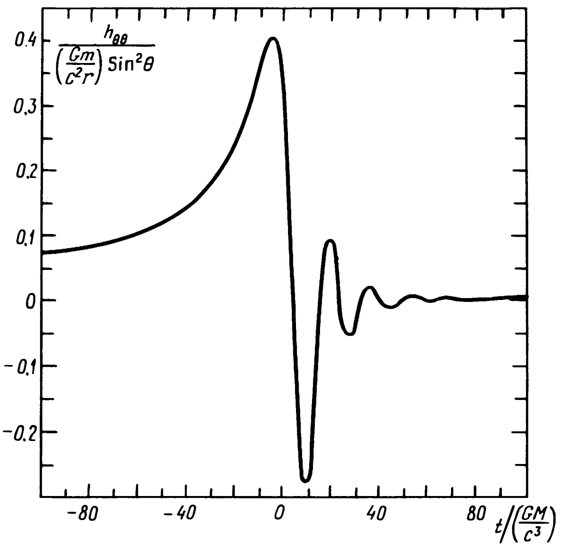


Fig. 20. The waveform for $l = 2$ of the gravitational radiation of a radially falling particle with v_∞ 0. The direction $\theta = 0$ coincides with the trajectory of the particle [according to Petrich *et al.* (1985)]. The sign of h in this and other similar figures depends on the definition; different authors use different definitions, which is of no great significance.

3.3. Gravitational Radiation of a Test Particle in the Field of a Black Hole

One of the simplest types of perturbation is the motion of a test particle of mass $m \ll M$ in the gravitational field of a black hole. Throughout this section, we invariably assume m/M to be so small that the back-reaction of the radiation on the motion of the particle is negligible.

First we consider the gravitational radiation of a particle that radially falls on the black hole at a parabolic velocity ($v_\infty = 0$). The results of numerical modeling of Davis *et al.* (1971, 1972) are plotted in Figure 19 and 20; see also Petrich *et al.* (1985).

Figure 19 shows the spectrum of the gravitational radiation averaged over all directions and measured by a distant observer. The total amount of emitted energy is $\Delta E = 0.01\, mc^2(m/M)$.

Figure 20 plots the field of the gravitational wave (the transverse components of metric perturbations) as a function of time at fixed r. The figure clearly shows how a burst of radiation coming directly from the falling particle is gradually transformed (in the neighborhood of $t = 0$) into the radiation from quasinormal modes of oscillation (subsequent decaying oscillations on the curve). The 'tails' (which must

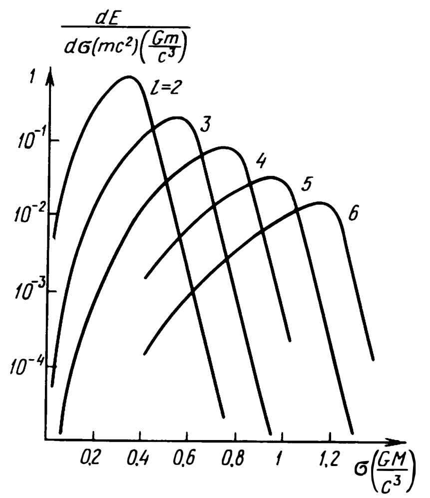


Fig. 21. The spectrum of total gravitational radiation of a particle with $v_\infty = 0$ and $\tilde{L} = 1.75$, for different modes.

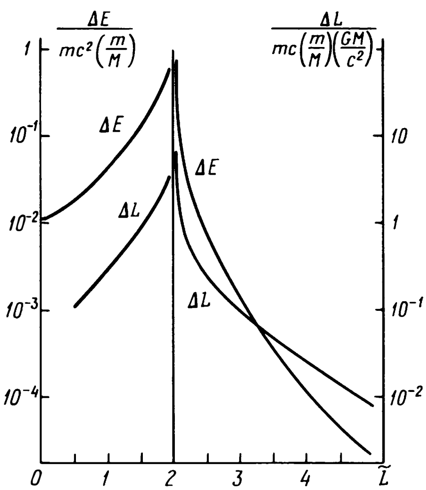


Fig. 22. Total radiated energy ΔE and total radiated angular momentum ΔL of a particle with $v_\infty = 0$.

dominate at $t \to \infty$) have amplitudes that are too small to be obtained in this numerical modelling.

Although the radiation coming directly from the source cannot be stringently separated from 'ringing', the shape of the oscillations indicates that this boundary lies in the neighborhood of $t \approx 0$. It can be concluded that the main fraction of the energy emitted as a result of the falling of a particle is emitted as the 'ringing radiation' in quasinormal modes.

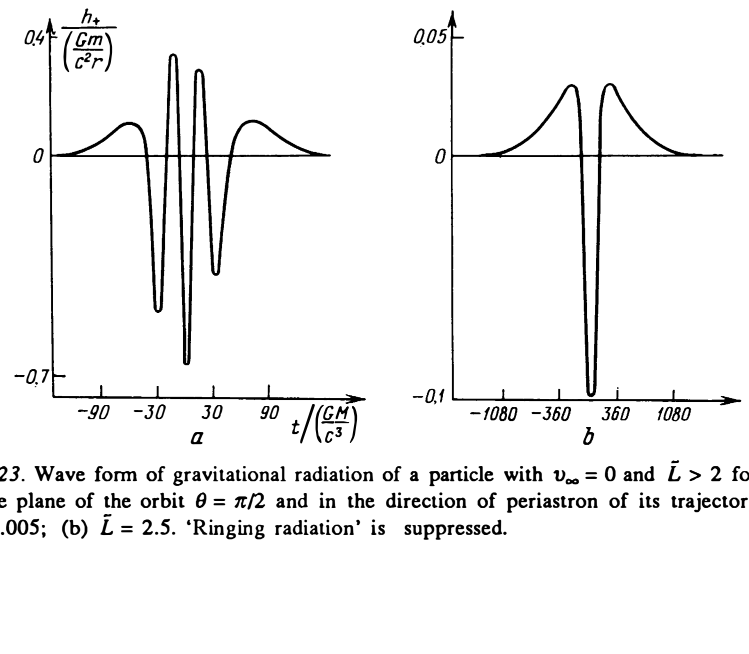


Fig. 23. Wave form of gravitational radiation of a particle with $v_\infty = 0$ and $\tilde{L} > 2$ for an observer in the plane of the orbit $\theta = \pi/2$ and in the direction of periastron of its trajectory, $\phi = 0$; (a) $\tilde{L} = 2.005$; (b) $\tilde{L} = 2.5$. 'Ringing radiation' is suppressed.

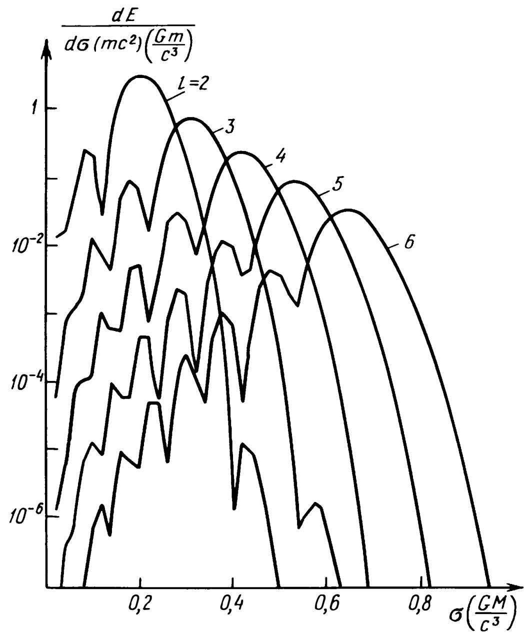


Fig. 24. Spectrum of radiation by a particle with $\tilde{L} = 2.005$ and $v_\infty = 0$.

Let us consider the fall of the particle possessing an angular momentum $\tilde{L}$ and parabolic velocity [Detweiler and Szedenitz (1979), Oohara and Nakamura (1983a, b)]. Figure 21 plots the energy spectrum of total radiation for $\tilde{L} = 1.75$ and Figure 22 plots the total emitted energy ΔE and the angular momentum ΔL as functions of $\tilde{L}$. Recall that as $\tilde{L} \to 2$, a geodesic orbit goes through many revolutions around the black hole, increasing thereby the emitted energy and angular momentum. For $\tilde{L} > 2$, the emitted energy and angular momentum again decrease, and the particle is not captured by the black hole but escapes to infinity (see Chapter 2). The greater $\tilde{L}$, the further from the $r = r_g$ circle the particle passes (Figure 17).

We need to emphasize the following fact mentioned in the papers cited above. If a particle falls into a black hole, quasinormal modes of oscillations are excited and the main part of energy is emitted precisely in the 'ringing radiation' due to these oscillations. If the particle is not captured into the black hole and escapes to infinity (i.e., is scattered by the black hole) and its energy at infinity is not large (i.e., is not relativistic), the quasinormal modes of the oscillation of the black hole are not excited.

This is clearly seen in Figure 23a and especially so in Figure 23b, where the perturbations of the gravitational field seem to be quite symmetric and we see nothing similar to the exponentially decaying modes of 'ringing' radiation that we recognize so clearly in Figure 20. The radiation spectrum shown in Figure 24 for each l-th harmonic points to the same behavior. When quasinormal modes are excited, the spectral maximum of each harmonic is largely determined by the mode

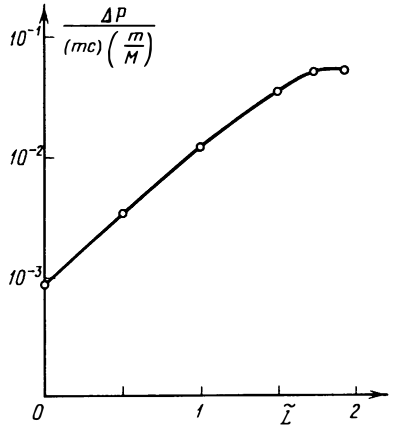


Fig. 25. Radiation of momentum, ΔP, in the fall of a particle with $v_\infty = 0$ as a function of $\tilde{L}$.

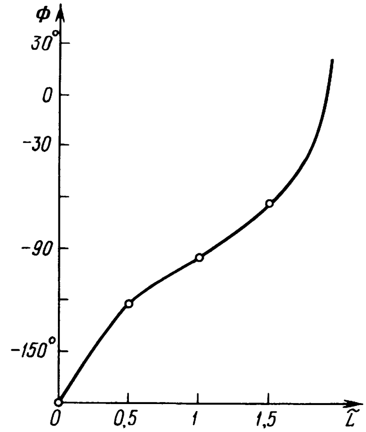


Fig. 26. Angle Φ as a function of $\tilde{L}$ (see text).

eigenfrequency and is independent of $\tilde{L}$. In the case of scattering ($\tilde{L} > 2$), the position of the maximum of each mode is a function of $\tilde{L}$. The position of the maximum of the radiation as a whole is determined by the double angular frequency of the motion of the particle at the periastron. Taken together, this means that 'ringing' is practically absent.

The physical interpretation of the absence of 'ringing' in the case of scattering of a particle by a black hole, is as follows. The periastron of the orbit of the scattered particle lies beyond the potential barrier located at $r \approx 1.5\ r_g$ because $r > \sqrt{2}r_g$. Correspondingly, the perturbations due to the particle itself do not excite these modes. But the gravitational radiation of such particles has the wavelength $\lambda \gg r_g$ and hence, (see Section 2.2), is reflected off the potential barrier, does not penetrate

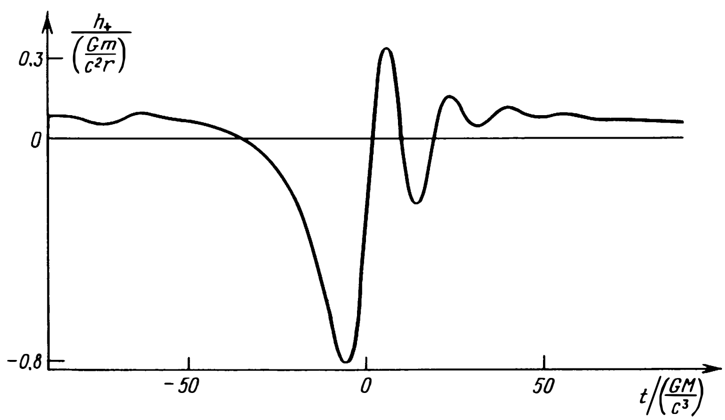


Fig. 27. Waveform of gravitational radiation for $\theta = \pi/2$ and $\phi = 0$ for a particle with $(1-v_\infty^2/c^2)^{-1/2} = 2$ and $\tilde{L} = 6.25$.

closer to the black hole, and cannot excite quasinormal modes. This is why a very good approximation for calculating the gravitational radiation of scattered particles is the standard formula [Landau and Lifshitz (1975)] which is valid for particles moving at $v \ll c$ in a flat space and ignores quasinormal modes.

The emission of gravitational waves by a falling particle is characterized by angular asymmetry. The waves carry off some of the momentum of the system. Figure 25 plots the radiated momentum as a function of angular momentum of the falling particle, and Figure 26 plots the angle Φ between the direction of the trajectory coming from infinity and that of the total momentum carried away by gravitational waves as a function of L.

Finally, let us consider the gravitational radiation emitted as a result of scattering of an ultrarelativistic particle (v_∞ on the order of c) by a black hole. Oohara demonstrated (1983) that, in this case, the quasinormal modes of oscillation of a black hole are excited. Two factors stimulate this excitation.

First, the periastron of the trajectory of such a particle may lie closer to the black hole than in the case of a scattering with $v_\infty = 0$ (see Chapter 2). It may lie even closer than the maximum of the potential barrier. Then the incoming wave will excite oscillations at eigenfrequencies.

Second, an ultrarelativistic particle emits gravitational waves at frequencies much higher than frequency of motion at the periastron [the effect of gravitational synchrotron radiation; see, e.g., Doroshkevich *et al.* (1972), Ruffini (1973), Chrzanowski and Misner (1974), and Ternov *et al.* (1975)]. High-frequency waves with $\lambda < r_g$ can penetrate through the barrier and excite the oscillation of quasinormal modes.

As an example, Figure 27 plots the result of Oohara (1983) for the perturbations of the metric in the gravity wave for a particle with $(1 - v_\infty^2/c^2)^{-1/2}$ and $\tilde{L} = 6.25$. The right-hand side of the graph clearly shows the exponentially decaying 'ringing' radiation from quasinormal modes.

In conclusion, we give a reference for the papers of Shapiro and Wasserman (1982) and Petrich *et al.* (1985) which deal with the emission from extended sources falling on a nonrotating black hole.

3.4. Power-Law 'Tails' of the Gravitational Radiation

Let us now look at the asymptotic behavior of the perturbed field of a black hole tending to nonperturbed state as $t \to \infty$ [Price (1972a, b), Thorne (1972), Leaver (1985, 1986a, 1986b)].

This behavior is determined by the following processes. Let a source of perturbations fall toward the black hole boundary. For instance, it could be a particle falling on the black hole, or 'rippling' on the surface of a contracting spherical cloud giving rise to a black hole.

The perturbations will again be analyzed by the method described in Section 3.2. Our problem is to analyze the asymptotic behavior of the function Φ at large values of time as $t \to \infty$. The perturbation source approaches the black hole boundary [i.e.

moves towards $r_* = -\infty$; see (3.2.1)] at a velocity tending to c (see Chapter 2). This means that as $t \to \infty$, all processes in the perturbation source must 'freeze out' for the observer who is at rest in the Schwarzschild reference frame, just as they are frozen on the surface of a collapsing star (see Chapter 2). The field Φ must also tend to a constant value on the source. It can be shown that this freezing obeys the law (for any multipole of order l)

$$\Phi_l = Q_0 + Q_1 \exp\left(\frac{r_* - ct}{2r_g}\right), \tag{3.4.1}$$

where Q_0 and Q_1 are constants. The characteristic time of the exponentially decaying term is of order r_g/c. Consequently, the waves Φ_l having this frequency are partly reflected by the potential barrier and returned to the black hole (to $r_* = -\infty$) and partly pass through the barrier and escape to infinity ($r_* = \infty$). The constant term, Q_0, generates a perturbation of infinite wavelength which is completely reflected by the barrier and cannot reach the external observer. As a result, only exponentially decaying waves can reach the observer directly from the perturbation source. But the damping of the entire radiation will not be exponential, because it involves the scattering of primary waves by the 'tails' of the potential barrier (i.e., by the space curvature). Before going into details of this effect, let us look carefully at the transmission of waves through potential barrier [Thorne (1972)].

Let a wave be emitted outward from the black hole (from $r_* = -\infty$ to $r_* = \infty$). We expand $\Phi_l\,(t, r_*)$ into the Fourier integral

$$\Phi_l(t, r_*) = \int_{-\infty}^{\infty} A(k) R^l_k(r_*)\, e^{-ikt}\, dk. \tag{3.4.2}$$

This wave is partly reflected and partly transmitted through the barrier. Equations (3.2.2) and (3.1.2) can be shown to imply that the general solution for this wave has the following asymptotic form for R^l_k:

$$R^l_k = e^{ikr_*} + \Gamma^{(R)}_k e^{-ikr_*}, \quad \text{as } r_* \to -\infty, \tag{3.4.3}$$

$$R^l_k = T^{(R)}_k e^{ikr_*}, \quad \text{as } r_* \to \infty, \tag{3.4.4}$$

where $\Gamma^{(R)}_k$ is the reflection coefficient and $T^{(R)}_k$ is the transmission coefficient of the rightward-travelling wave, that is, propagating from $r_* = -\infty$ to $r_* = \infty$ [this is indicated by the superscript (R)]. For small $|k| \ll 1/r_g$, the coefficients are

$$\Gamma^{(R)}_k = -1 + \alpha r_g ik, \tag{3.4.5}$$

$$T^{(R)}_k = \frac{\beta}{(2l-1)!!}(r_g ik)^{l+1}. \tag{3.4.6}$$

Here α and β are constants on the order of unity. As $k \to 0$, the waves undergo total reflection from the barrier.

For leftward-travelling waves [from $r_* = \infty$ to $r_* = -\infty$, superscript (L)], one arrives at a similar solution, with coefficients

$$\Gamma_k^{(L)} = (-1)^{l+1} + \frac{\gamma}{(2l-1)!!}(r_g ik)^{2l+1}, \tag{3.4.7}$$

$$T_k^{(L)} = \delta(r_g ik)^{l+1}; \tag{3.4.8}$$

γ and δ are constants on the order of unity. As $k \to 0$, we again come to the total reflection from the barrier.

Now we turn to wave damping at $t \to \infty$. Let us consider the region of $r > 1.5r_g$, that is, the region beyond the potential barrier. The waves Φ from the source which were partly transmitted through the barrier are scattered backward at the 'tail' of the potential at $r_* \ll r_g$. These waves reach $r_* \approx 0$, are again reflected by the potential barrier, and interfere with the waves propagating backwards. The law of damping is dictated by backscattering and interference. The result is

$$\Phi_l \sim t^{-(2l+2)}. \tag{3.4.9}$$

Waves scattered by the 'tail' of the potential barrier also penetrate into the region $r_g < r < 1.5r_g$. As a result, the same formula (3.4.9) describes damping in this region as well.

The asymptotic behavior of the decay of radiative multipole of perturbations ($l \geq 2$) of the black-hole gravitational field as $t \to \infty$ is thus described by the formula (3.4.9). Recall that a field perturbation with $l = 1$(it corresponds to angular momentum) does not change at all (this is a nonradiative mode). All other multipoles of perturbations (i.e. those with $l \geq 2$) completely vanish as $t \to \infty$.

The collapse of a slightly asymmetric nonrotating body first produces a slightly perturbed black hole (with slightly perturbed boundary and perturbed external field) but these perturbations are all radiated off (partly into and partly out of the black hole). At $t \to \infty$, the black hole is described exactly by the Schwarzschild metric.

The same conclusion holds for fields with spin different from two. All radiation multipoles of such fields ($l \geq s$) are also radiated away and the asymptotic behavior of damping is described by the same formula (3.4.9). Price put this conclusion in the following succinct form: "All things that can be radiated get radiated away completely ('what is permitted is compulsory')."

It should be mentioned in conclusion that the inference on the inevitable radiation of radiative modes of the $s = 1$ field (electromagentic field) was obtained by Ginzburg (1964) and Ginzburg and Ozernoy (1964), and for the $s = 2$ (gravitational) field by Doroshkevich *et al.* (1965) and Novikov (1969).

3.5. Cross-Section of Wave Scattering by a Black Hole

Consider the scattering of plane waves coming from infinity by a black hole [Handler and Matzner (1980)].

Recall that scattering is described by the so-called differential-scattering cross-section. For example, the differential scattering cross-section $d\sigma$* in the classical problem of scattering of a pencil of parallel rays is given by the following expression in the geometrical optics approximation

$$d\sigma = b\frac{db}{d\theta}(\sin\theta)^{-1}\, d\Omega, \tag{3.5.1}$$

where $b=b\ (\theta)$ is the impact parameter of the ray deflected in the field off the scattering center by an angle θ off the direction of incidence, and $d\Omega$ is an element of the solid angle ($d\Omega = 2\pi \sin\theta\, d\theta$).

In the case of a black hole and in the limit of the geometrical optics approximation, the dependence $b = b(\theta)$ is given by equations (2.8.6)–(2.8.7). The calculation of $d\sigma$ is then trivial. The wave problem, in which it is important to account for the interference of scattered waves and for the absorption of a part of the waves by the black hole, is of special interest. This problem is solved by expanding the incident waves in spherical harmonics and then analyzing the scattering of each of them by the potential barrier (see Section 3.1) and summing up the results.

First of all we follow the Handler–Matzner paper (1980) and review the qualitative features that can be expected for the differential cross-section.

The first such feature is obvious and concerns the forward scattering ($\theta \approx 0$). When a wave is scattered by a small angle, this situation corresponds in the semiclassical approximation to the passage of the wave at a large distance from the black hole (large b or large l in the spherical harmonic expansion) so that scattering occurs on the Coulomb potential. In this case the beam deviation is $\theta \propto 1/b$. Substituting this relation into (3.5.1), we obtain

$$d\sigma/d\Omega \propto 1/\theta^4.$$

Other features are implied by the existence in the Schwarzschild field of a circular orbit for massless particles and by specific conditions created for the interference of waves by scattering to angles close to $\theta = \pi$. The last case, that is, the effect of backscattering, is of special importance and interest. It was given the name 'glory'. Matzner *et al.* (1985) analyzed this phenomenon for the Schwarzschild black hole. Briefly, the effect is as follows.

Figure 28 plots the ray reflected exactly by a $\theta = \pi$ by the solid curve. For a black hole of mass M, the corresponding impact parameter b_{gl} is

$$b_{gl} \approx 5.35\, \tilde{M}. \tag{3.5.2}$$

Here we have denoted $\tilde{M} \equiv GM/c^2$. The rays with b slightly different from b_{gl}, bypassing the black hole on opposite sides and then propagating after scattering parallel to each other in a direction close to $\theta = \pi$, are also shown in this figure.

* We hope that the reader does not confuse here the notation σ for scattering cross-section with frequency σ of p. 39, since here we consider real frequencies denoted by ω.

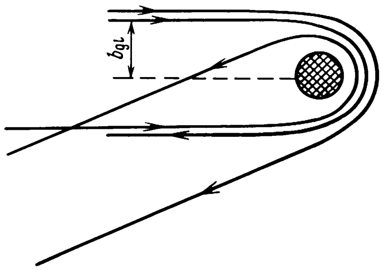


Fig. 28. Trajectories of rays with impact parameters close to b_{gl} ('glory' interference).

The path difference between rays is small and the resulting phase difference produces interference.

Matzner *et al.* (1985) solved the problem of 'glory' scattering using the generalized WKB approximation. Their result is valid for $\tilde{M}\,\omega c^{-1} \ll 1$ (ω is the frequency of the wave) and $|\theta - \pi| \ll 1$. In this situation they find

$$\frac{d\sigma}{d\Omega} = 4\pi^2\lambda^{-1}b_{gl}^2\left|\frac{db}{d\theta}\right|^2 J_{2s}^2(2\pi\lambda^{-1}b_{gl}\sin\theta), \tag{3.5.3}$$

where λ is the wavelength, s is the spin of the wave, and J_{2s} is the Bessel function of order $2s$.

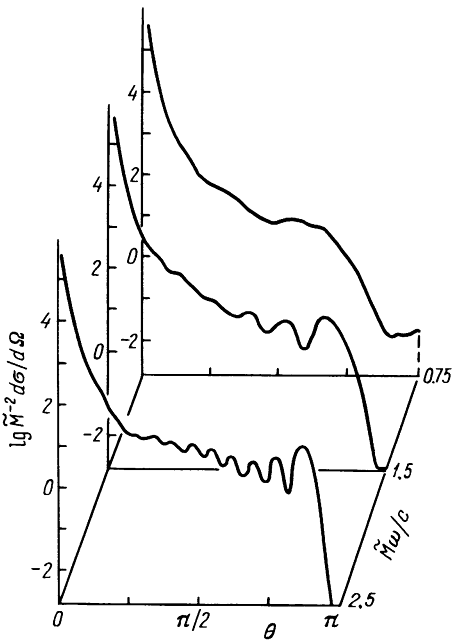


Fig. 29. Differential cross-section of the scattering of gravitational radiation by a nonrotating black hole for different frequencies ω.

Substituting into (3.5.3) the numerical values $b_{gl} \approx 5.35\tilde{M}$ and $\mathrm{d}b/\mathrm{d}\theta|_{\theta=\pi} \approx -0.15\tilde{M}$, we obtain

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \approx 8.58\pi\omega\tilde{M}^3 c^{-1} J_{2s}^2(5.35\omega\tilde{M}c^{-1} \sin\theta). \tag{3.5.4}$$

If the conditions $\tilde{M}\omega\, c^{-1} \gg 1$ and $|\theta - \pi| \ll 1$ are not satisfied, the scattering cross-section is not given by a simple formula and is found numerically. Figure 29 plots the results of computations of Handler and Matzner (1980) for the scattering of gravitational waves ($s = 2$) with $\tilde{M}\omega c^{-1} = 0.75$, 1.5, and 2.5. The $\mathrm{d}\sigma/\mathrm{d}\Omega \sim \theta^{-4}$ singularity in forward scattering ($\theta \approx 0$) is apparent on all curves.

The interference pattern of 'glory' backscattering ($\theta = \pi$) is already clearly seen for $\tilde{M}\omega c^{-1} = 2.5$. A comparison of the results of numerical modeling with formula (3.5.4) for $\tilde{M}\omega\, c^{-1} = 2.5$ indicates a good fit in the position and width of the 'glory' maximum closest to $\theta = \pi$. As frequency ω increases, the minima of curves of the $\theta = \pi$ backscattering become progressively deeper. This effect is caused by enhanced absorption of waves with small l (that is, small b) at increasing frequencies. In the $\omega \to \infty$ limit, the $\theta = \pi$ backscattering of gravitational waves (or any waves with $s \neq 0$) gives $\mathrm{d}\sigma/\mathrm{d}\Omega = 0$.

The total absorption cross-sections for $\tilde{M}/\omega c^{-1} = 0.75$, 1.5, and 2.5 are $72.4\tilde{M}^2$, $83.36\tilde{M}^2$, and $83.61\tilde{M}^2$, respectively. Recall that for $\omega \to \infty$ the absorption cross-section is $27\pi\tilde{M}^2 \approx 84.81\tilde{M}^2$.

On the frequency-dependence of the scattering cross-section of waves on a Schwarzschild black hole, see also Sanchez (1976, 1977, 1978a, b).

Chapter 4

Rotating Black Hole

4.1. Formation of a Rotating Black Hole

In the preceding chapters we have demonstrated that the gravitational collapse of a spherical nonrotating mass produces a spherically symmetric black hole when the radius of the body becomes less than the gravitational radius. In Section 3.4 we have shown that after a black hole has been formed in the collapse of a body slightly deviating from spherical symmetry, all deviations from spherical symmetry rapidly vanish, except those due to small angular momentum J. Angular momentum remains practically unaltered through the collapse. If the collapsing body has an electric charge, then its total charge and electric field due to this charge are also unaltered, while all other components of the electromagnetic field in the external space also rapidly vanish.

Suppose the collapsing body deviates considerably from spherical symmetry and its angular momentum and electromagnetic field are large. Will a black hole form? If it does, what will its properties be?

It will be argued in subsequent chapters that the contraction of an arbitrary rotating mass, possessing an electromagnetic field to a sufficiently small size, produces a black hole, and that all the properties of this black hole and its external gravitational fields are completely determined by three parameters: mass M, angular momentum J, and electric charge Q. [1] The other properties of the collapsing body, such as its composition, asymmetry in the distribution of mass and electric charge, the magnetic field and its characteristics, and so forth, do not influence the properties of the resultant black hole.

This conclusion follows qualitatively from an analysis of the behavior of small perturbations in the course of the formation of the spherical black hole (Section 3.4). Radiative multipoles of all fields rapidly vanish, and only nonradiative modes survive, being determined by the three parameters M, J, Q. Physics knows no other 'classical' physical fields with other (nonradiative) modes. Gravitational radiation carries off a part of the energy and angular momentum of the collapsing mass in the course of the gravitational collapse when the deviations from symmetry are large. As a result, M and J of the black hole become slightly smaller than those the body had before the collapse (this will be discussed later). This reduction could not be found in the analysis of small perturbations because the back-reaction of the

perturbations on the metric was assumed to be negligible. In astrophysics, the total electric charge of a body can typically be treated as small and accordingly neglected. Therefore, we first consider the case of $Q = 0$. The case of nonzero charge is analyzed in Section 4.8.

What is the gravitational field of a black hole with nonzero angular momentum J? It is shown in Section 6.4 that this field is described by the stationary axisymmetric solution of Einstein's equations which was found by Kerr (1963). We begin by describing the physical properties of the external space of a rotating black hole.

4.2. The '3 + 1' Split of Spacetime Outside the Black Hole

The external field of a nonrotating black hole (the Schwarzschild field) and the specifics of motion of particles in this field were investigated in Chapter 2 using the Schwarzschild reference frame. This reference frame is static, independent of time, and uniquely defined for each black hole. [2] It can be thought of as a lattice 'welded' of weightless rigid rods. The motion of particles was defined with respect to this lattice. For the time variable, we used the time t of an observer placed at infinity. True, the rate of the flow of the physical (proper) time τ at each point of our lattice did not coincide with that of t (time is slowed down in the neighborhood of a black hole), but this 'parametrization' in t = const meant simultaneity in our entire frame of reference.

The Schwarzschild reference frame in a certain sense resembles the absolute Newtonian space in which objects move, and t resembles the absolute Newtonian time of the equations of motion.

Of course, important differences exist. Our 'absolute' space is curved (curved very strongly close to the black hole) and the 'time' t is not the physical time.

This reference frame is used not only to facilitate mathematical manipulations in solving, say, the equations of motion, but also to increase the graphical clarity. We make use of the habitual concepts of the Newtonian physics (the 'absolute' rigid space as the scene on which events take place, and the unified time) and thereby help our intuition. Although the Schwarzschild reference frame has a singularity at r_g we choose this reference frame for the spacetime outside the black hole and not, say, the Lemaitre frame which has no singularity at r_g but is everywhere deformable.

Obviously, a rigid reference frame can be chosen only because the spacetime outside the black hole is static. In the general case of a variable gravitational field, this choice is impossible since the spatial grid would be deformed with time.

In the case of a rotating black hole (Kerr metric), the spacetime outside it is stationary and one can choose a time-independent reference frame which asymptotically transforms in the Lorentz frame at infinity. The Boyer–Lindquist coordinates (1967) represent such a frame of reference. Let us express the Kerr metric in these coordinates:

$$ds^2 = -\frac{\rho^2\Delta}{A}c^2\,dt^2 + \frac{A\sin^2\theta}{\rho^2}\left(d\phi - \frac{2aGMr}{c^2A}dt\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2\,d\theta^2, \qquad (4.2.1)$$

where

$$\rho^2 \equiv r^2 + \frac{a^2\cos^2\theta}{c^2}, \quad \Delta \equiv r^2 - \frac{2GMr}{c^2} + \frac{a^2}{c^2},$$

$$A = \left(r^2 + \frac{a^2}{c^2}\right)^2 - \frac{a^2\Delta\sin^2\theta}{c^2}; \qquad (4.2.2)$$

a is the specific angular momentum ($a = J/M$) and M is the black hole mass. In what follows we use the system of units in which $c = G = 1$. Presumably, the physically meaningful solutions are those with $M^2 > a^2$ (see note [1] to p. 51).

The properties of the three-dimensional space t = const in the (4.2.1) metric, which is external withe respect to the black hole, do not change with time. This means that there exists a Killing vector field (see Appendix) directed along the lines of time t; shifting the spatial section along this field, we pass from one section to another identical to it. We can thus 'trace' in the space a grid which remains invariant in the transition from one section to another along the Killing vector field.[3] The variable t, that is, the time of the distant observers, can serve as the universal 'time' enumerating the spatial sections, as was the case for the Schwarzschild spacetime.

Important differences must be mentioned, of course.

(1) In the case of the Schwarzschild field, the transition from one three-dimensional section to another, preserving the coordinate grid, is carried out by shifting along the time lines perpendicular to the spatial section. The situation in the Kerr field is different, the Killing vector field being tilted with respect to the section t = const; the tilting angle is different for different r and θ.

(2) The Killing vector that realizes the transition from one section to another becomes space-like at points close to the boundary of the black hole [see (p. 56)]. This means that a three-dimensional rigid grid *cannot be made, in such regions, of material bodies* (cannot be 'welded' of rods). In the neighborhood of a black hole, this grid would move at superluminal velocity with respect to any observer (on a time-like world line).

Despite these specifics, we can still operate with our space sections t = const as with 'absolute' rigid space (resembling the Newtonian case) and with t, as with 'time' which is universal in the entire 'space' (of course, subject to all the qualifications given above).

In general relativity, the splitting of spacetime in an arbitrary gravitational field into a family of three-dimensional spatial sections (in general, their geometries vary from section to section) and the universal 'time' that enumerates these sections, is referred to as the '3+1' split of spacetime,[4] or the kinematic method [Vladimirov (1982)]. This method is especially useful when all spatial sections are identical and

the motion of particles, electromagnetic processes, etc., that unfold on this invariant 'scene' can be described in terms of a universal 'time' t. We have already mentioned that in this case our intuition is supported by our 'visual' images of space and time supplied by everyday experience.

Studying the processes in the vicinity of stationary black holes, we employ the kinematic method. As spatial sections, we choose the t = const section in (4.2.1) metric; t is the time coordinate.

4.3. Chronometric Reference Frame and a Reference Frame of Locally Nonrotating Observers

First we consider the geometric properties of our 'absolute' space. They are described by a tree-dimensional metric obtained from (4.2.1) by setting $dt = 0$. In this three-dimensional 'absolute' space we can, at a fixed moment of the universal 'time' t = const, analyze the distribution of three-dimensional vector fields, or calculate, say, the three-dimensional divergence of the vector field **A**, and so on. The change in **A** with 'time' t at a fixed point of the 'absolute' space is given by the derivative $\partial \mathbf{A}/\partial t$.

Consider now the reference frame of the observers which are at rest in the 'absolute' space t = const, that is, observers who 'sit still' on our rigid nondeformable lattice. This frame of reference is called the chronometric [Vladimirov (1982)], Lagrange [Thorne and Macdonald (1982), Macdonald and Thorne (1982), Thorne, Price, and Macdonald (1986)] or Killing reference frame. Let us look at the forces acting in this frame owing to the presence of a rotating black hole.

The three-dimensional components of the acceleration vector $\tilde{F}_i$ in the coordinates r, θ, ϕ [acceleration of 'free fall'; see (A.61)] are given by the expressions [Vladimirov (1982)]

$$\tilde{F}_r = \frac{M(\rho^2 - 2r^2)}{\rho^2(\rho^2 - 2Mr)}, \quad \tilde{F}_\theta = \frac{Mra^2 \sin 2\theta}{\rho^2(\rho^2 - 2Mr)}, \quad \tilde{F}_\phi = 0. \tag{4.3.1}$$

We mark all quantities in this chronometric reference frame by a tilde to avoid confusing them with the quantities used hereafter.

The physical components of acceleration are [5]

$$\tilde{F}_{\hat{r}} = \frac{M(\rho^2 - 2r^2)\sqrt{\Delta}}{\rho^3(\rho^2 - 2Mr)}, \quad \tilde{F}_{\hat{\theta}} = \frac{Mra^2 \sin 2\theta}{\rho^3(\rho^2 - 2Mr)}, \quad \tilde{F}_{\hat{\phi}} = 0. \tag{4.3.2}$$

The reference frame of our observers is rigid, so that the deformation rate tensor vanishes [see (A.60)]:

$$\tilde{D}_{ik} = 0. \tag{4.3.3}$$

The angular velocity tensor (A.59) is

$$\tilde{A}_{r\phi} = -\frac{Ma(\rho^2 - 2r^2)\sin^2\theta}{\rho(\rho^2 - 2Mr)^{3/2}}, \quad \tilde{A}_{\theta\phi} = -\frac{Mra\sin 2\theta}{\rho(\rho^2 - 2Mr)^{3/2}}, \quad \tilde{A}_{r\theta} = 0. \qquad (4.3.4)$$

The nonvanishing tensor $\tilde{A}_{ik}$ signifies that gyroscopes that are at rest in our reference frame are precessing with respect to it and, hence, with respect to distant objects, because at a large distance our rigid reference frame becomes Lorentzian. The tensor $\tilde{A}_{ik}$ is proportional to the specific angular momentum of the black hole and reflects the presence of the 'vortex' gravitational field due to its rotation.

The following important difference between the external fields of a rotating and a nonrotating black holes must be emphasized.

If a black hole is not rotating, the condition $t =$ const signifies physical simultaneity in the entire external space for the observers that are at rest in it (with respect to a rigid reference frame). In the case of a rotating black hole, a nonvanishing component g_{0i} in the rigid reference frame forbids [see Landau and Lifshitz (1975)] the introduction of the concept of simultaneity. Usually, the events with equal t are said to be simultaneous in the time of a distant observer. But this does not mean at all the physical simultaneity of these events which is determined by the synchronization of clocks via sending and receiving light signals.

Note that the components $\tilde{F}_r$, $\tilde{F}_\theta$, the components $\tilde{A}_{ik}$, and the angular velocity of precession $\tilde{\Omega}_{pr}$ of a gyroscope, calculated using these components [see (A.62)], tend to infinity, while the component g_{oo} in (4.2.1) (which determines the rate of flow of time) vanishes at

$$\rho^2 - 2Mr = r^2 + a^2\cos^2\theta - 2Mr = 0 \qquad (4.3.5)$$

or at $r = r_1$, where r_1 is given by the relation

$$r_1 = M + \sqrt{M^2 - a^2\cos^2\theta}\,. \qquad (4.3.6)$$

These properties signified that a *physical singularity* exists at this point in the reference frame, and this frame cannot be prolonged closer to the black hole, that is, observers cannot be at rest relative to our grid. [6] Formally, the reason for this is the same as in the Schwarzschild field at $r = r_g$. Namely, the world line of the observer, $r =$ const, $\theta =$ const, $\phi =$ const, ceases to be time-like, as indicated by the reversal of sign of g_{oo} at $r < r_1$. In fact, an essential difference in comparison with the Schwarzschild field must be emphasized.

In order to obtain a word line inside the light cone in a nonrotating black hole for $r < r_l$, it was sufficient to perform the transformation

$$r = r(\tilde{r}, t), \quad \frac{\partial r}{\partial t} = A_1 \neq 0. \qquad (4.3.7)$$

With a suitable choice of $A_1 = A_1(r)$, the $\tilde{r} =$ const, $\phi =$ const, $\theta =$ const line becomes time-like. This means that at $r < r_g$, a body necessarily moves centerward *along the radius*, and that r_g is the boundary of an isolated black hole.

In the case of a rotating black hole [we assume $\Delta > 0$; see (4.2.1)] a transformation of the type of (4.3.7) cannot generate a time-like world line. But the transformation of the type

$$\phi = \phi(\tilde{\phi}, r, \theta, t), \quad \frac{\partial \phi}{\partial t} = \Omega_1 \neq 0 \tag{4.3.8}$$

makes this possible (Ω_1 is a function of r and θ). This fact signifies that if $r < r_1$ and $\Delta > 0$, all bodies necessarily participate in the rotation around the black hole (the sense of rotation is determined by the sign of a; see below) with respect to a rigid coordinate grid that stretches to infinity. As for the motion along the radius r, bodies can move in the range $r < r_1$, $\Delta > 0$ both increasing and decreasing the value of r.

Therefore, the *static (stationary) limit* r_1 has quite different nature for a rotating black hole than in the Schwarzschild field. Inside it, bodies are unavoidably dragged into rotation, although r_1 is not the event horizon because a body can escape from this region. In metric (4.2.1), the event horizon lies at $\Delta = 0$, that is, at $r = r_+$, where

$$r_+ = M + \sqrt{M^2 - a^2}\,. \tag{4.3.9}$$

The region $r_+ < r < r_1$ is called the *ergosphere*.

A rigid, static frame of reference (i.e., one that is at rest relative to a distant observer) made of material bodies does not stretch to r_+. The static limit lies beyond the horizon and coincides with it at the pole (Figure 30). An important feature of a static reference frame is the precession of gyroscopes in it, as we have mentioned above. Our reference frame rotates at each of its points with respect to the local Lorentz frame. This is, of course, a reflection of the fact that the rotation of the black hole changes the state of motion of local Lorentz frames, dragging them into the rotation around the black hole. This effect has been known qualitatively for quite a long time for the case of the weak gravitational field of a rotating body [Thirring and Lense (1918)].

Now we introduce into the external space of a rotating black hole a reference frame which does not rotate, in the sense given above, with respect to the local Lorentz frame. This frame of reference is said to be the reference frame of locally nonrotating observers [also known as the reference frame of zero angular momentum

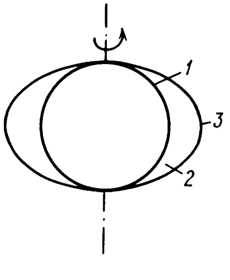


Fig. 30. A rotating black hole: 1 - horizon, 2 - ergosphere, 3 - static limit.

observers (ZAMO)]. Obviously, such a frame cannot be rigid. To introduce it, we trace a congruence of world lines which are everywhere orthogonal to the spatial sections $t = \text{const}$ that we chose. By definition, these time-like lines are not twisted and form the sought reference frame. Observers that are at rest in this frame are said to be locally nonrotating observers [sometimes this reference frame is referred to as Eulerian; see Thorne and Macdonald (1982)]. These observers move with respect to Boyer–Lindquist coordinate system, that is, they move in the 'absolute' space.[7] This motion takes place at constant r and θ, at a constant (in time) angular velocity in ϕ. If the angular velocity ω is determined with respect to the universal time t (time of the distant observer), then

$$\omega \equiv \frac{d\phi}{dt} = -\frac{g_{\phi t}}{g_{\phi\phi}} = \frac{2Mar}{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}, \tag{4.3.10}$$

where $g_{\phi t}$ and $g_{\phi\phi}$ are taken from (4.2.1).

If angular velocity is measured by the clock of the locally nonrotating observer, then

$$\Omega_\tau = \frac{\omega}{\sqrt{-g_{tt} - 2\omega g_{t\phi} - \omega^2 g_{\phi\phi}}}. \tag{4.3.11}$$

The linear physical velocity of locally nonrotating observers with respect to a rigid reference frame is

$$v_\phi = \frac{2Mra\sin\theta}{\rho^2\sqrt{\Delta}}. \tag{4.3.12}$$

As could be expected, this velocity becomes equal to the speed of light at the static limit $r = r_1$ and exceeds it in the ergosphere.

We again emphasize that the proper time of locally nonrotating observers, τ, is not equal to the universal 'time' t. The relation between them is equal by the 'lapse' function α:

$$\left(\frac{d\tau}{dt}\right)_{\text{ZAMO}} \equiv \alpha = \left(\frac{\rho^2\Delta}{A}\right)^{1/2}. \tag{4.3.13}$$

The expressions for the vector $\boldsymbol{F}$ of acceleration of free fall in the reference frame of locally nonrotating observers are:

$$F_r = \frac{M}{\rho^2\Delta\Delta_1}[(r^2+a^2)^2(a^2\cos^2\theta - r^2) + 4Mr^3a^2\sin^2\theta],$$
$$F_\theta = a^2\sin 2\theta\,\frac{Mr(r^2+a^2)}{\rho^2\Delta}, \tag{4.3.14}$$
$$F_\phi = 0,$$

where $\Delta_1 = \rho^2(r^2 + a^2) + 2Mra^2 \sin^2\theta$. This vector is related to α as follows:

$$F = -\nabla \ln \alpha. \tag{4.3.15}$$

The tensor of deformation rates of the reference frame is written in the form

$$\begin{aligned} &D_{rr} = D_{r\theta} = D_{\theta\theta} = D_{\phi\phi} = 0, \\ &D_{r\phi} = -Ma[2r^2(r^2 + a^2) + \rho^2(r^2 - a^2)] \sin^2\theta(\rho^3\sqrt{\Delta\Delta_1})^{-1}, \\ &D_{\theta\phi} = 2Mra^3 \sin^3\theta \cos\theta\sqrt{\Delta}\,(\rho^3\sqrt{\Delta_1})^{-1}, \end{aligned} \tag{4.3.16}$$

and the tensor $A_{ik} = 0$.

The introduced frame of reference has no singularities at the static limit and extends into the ergosphere up to the boundary of the black hole, $r = r_+$. At $r \leq r_+$, falling along r necessarily occurs in addition to the rotation around the black hole. At $r = r_+$, the reference frame of locally nonrotating observers has a physical singularity $F_r \to \infty$, as $r \to r_+$ [see formula (4.3.14)].

As we approach the event horizon, the angular velocity of rotation of locally nonrotating observers tends to a limit

$$\omega_+ = c^3 a/2GMr_+. \tag{4.3.17}$$

This limit is constant at the horizon, being independent of θ. It is called the angular velocity of rotation of the black hole (or horizon), Ω^H.

At the spatial infinity, the reference frame of locally nonrotating observers transforms into the same Lorentz frame as the Boyer–Lindquist coordinate system (the chronometric reference frame) does.

To conclude the section, consider the 'rotation' of locally nonrotating observers and the precession of gyroscopes in a reference frame fixed to these observers.

On the one hand, the reference frame of these observers was chosen to be nonrotating, that is, chosen such that $A_{ik} = 0$.

This means that there is no rotation of the reference frame with respect to the locally Lorentz frame and, hence, no precession of gyroscopes in the reference frame of locally nonrotating observers. On the other hand, it is said in, for example, the monograph of Misner, Thorne, and Wheeler (1973) that gyroscopes precess with respect to locally nonrotating observers at an angular velocity

$$\Omega_{pr} = \frac{1}{2}\sqrt{\frac{g_{\phi\phi}}{g_{tt} - \omega^2 g_{\phi\phi}}}\left[\frac{\omega_{,\theta}}{\rho} e_{\hat{r}} - \frac{\Delta^{1/2}\omega_{,r}}{\rho} e_{\hat{\theta}}\right], \tag{4.3.18}$$

where $e_{\hat{r}}$ and $e_{\hat{\theta}}$ are unit vectors along r and θ, respectively, and the quantities $g_{\alpha\beta}$ are taken from (4.2.1). Is it possible to make these statements compatible?

The paradox is solved in the following way. Recall that the motion of a small element of an arbitrary frame of reference with respect to a locally comoving

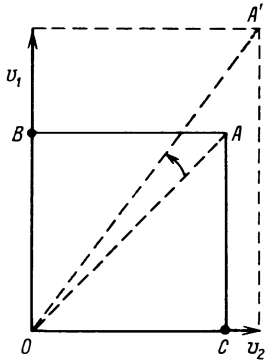


Fig. 31. Tilting of the diagonal OA owing to anisotropic deformation of a volume element along the directions OB and OC.

Lorentz frame consists of a rotation around the instantaneous rotation axis and of a deformation along the principal axes of the deformation rate tensor. In the case of no rotation ($A_{ik} = 0$), we have only deformation. A gyroscope whose center of mass is at rest in the reference frame, does not precess with respect to the principal axes of the deformation rate tensor. If lines comoving with the frame of reference (lines 'glued' to it) are traced along these directions, a gyroscope cannot change its orientation with respect to these lines. But this does not mean that the gyroscope does not change its orientation with respect to any line traced in a given element of volume in the comoving reference frame. Indeed, Figure 31 shows that anisotropic deformation tilts the lines traced, say, at an angle of 45° to the principal axes of the deformation tensor, so that they turn closer the direction of greatest extension. The gyroscope precesses with respect to these lines even though $A_{ik} = 0$. It is this situation that we find in the case of locally nonrotating observers in the Kerr metric.

Consider locally nonrotating observers in the equatorial plane. Everywhere $A_{ik} = 0$, and formulas (4.3.16) imply that only the component $D_{r\phi}$ is nonzero. This means that the instantaneous orientations of the principal axes of the deformation tensor lie at an angle of 45° to the vectors $e_{\hat{r}}$ and $e_{\hat{\phi}}$. Note that the coordinate lines are 'glued' to the reference frame. A gyroscope does not rotate with respect to the principal axes but, in view of the remark made above, does rotate relative to the ϕ coordinate line and, hence, relative to $e_{\hat{\phi}}$ (and therefore, relative to the vector $e_{\hat{r}}$, perpendicular to $e_{\hat{\phi}}$, which is not 'glued' to the reference frame; see below).

If a locally nonrotating observer always orients its vectors frame along the directions $e_{\hat{r}}$, $e_{\hat{\phi}}$, and $e_{\hat{\theta}}$ the gyroscope thus precesses with respect to this frame as given by formula (4.3.18), although in the observers frame of reference we have $A_{ik} = 0$. The $e_{\hat{r}}$, $e_{\hat{\phi}}$, $e_{\hat{\theta}}$ frame is a natural one; the precession of a gyroscope must be determined with respect to this frame because it is dictated by the symmetry of the space around the observer. But we could introduce a different frame as well, for example, a frame which is also fixed to locally nonrotating observers but does not rotate with respect to the instantaneously comoving Lorentz frame. Obviously, gyroscopes do not precess in such a frame.

Finally, we notice that if we choose, at some instant of time, one system of coordinate lines 'glued' to locally nonrotating observers and oriented strictly along r, and another system oriented along ϕ, the coordinate lines directed along ϕ slide with time in the 'absolute' space along themselves while the lines perpendicular to them 'wind' on the black hole and become helical, because they are dragged by a faster motion of the observers located closer to the black hole; hence, these lines rotate with respect to the ϕ lines.

4.4. Spacetime of a Rotating Black Hole

Here we consider the general properties of the spacetime of a rotating black hole described by the solution of (4.2.1).

Let us introduce a coordinate system which does not have coordinate singularities at the event horizon r_+ in the same manner as it was done in the Schwarzschild spacetime.[8] In that case we could use the world lines of photons moving centerward along the radii as coordinate lines [see (2.4.12)]. The world lines of photons moving toward a rotating black hole can also be chosen, but now the trajectories of the photons wind up on the black hole in its neighborhood, because they are dragged into rotation by the 'vortex' gravitational field. Therefore, if the black hole rotates, we have to supplement the substitution of coordinates [like (2.4.11)] with a 'twist' in the coordinate ϕ.

The simplest expression for the metric is obtained if we use the world lines of the photons that move at infinity at constant θ and whose projection of the angular momentum on the rotation axis of the black hole is $L_z = aE \sin^2 \theta$ (see the next section), where E is the particle energy at infinity. It can be shown that a transition to such a reference frame of 'freely falling' photons is achieved by a change of coordinates:

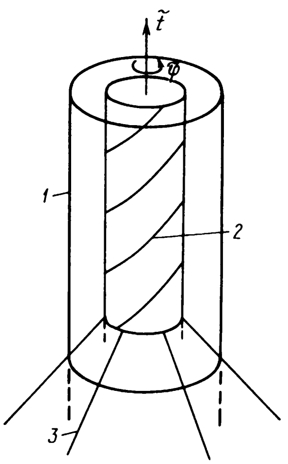


Fig. 32. Spacetime of a rotating black hole: 1 - null world line along the static limit, 2 - 'outgoing' photons forming the horizon, 3 - photons falling into the black hole.

$$\mathrm{d}\tilde{V} = \mathrm{d}t + (r^2 + a^2)\frac{\mathrm{d}r}{\Delta}, \quad \mathrm{d}\tilde{\phi} = \mathrm{d}\phi + a\frac{\mathrm{d}r}{\Delta}. \tag{4.4.1}$$

The resulting system of coordinates is known as the Kerr coordinate system (1963):

$$\begin{aligned} \mathrm{d}s^2 = &-[1 - \rho^{-2}(2Mr)]\,\mathrm{d}\tilde{V}^2 + 2\,\mathrm{d}r\tilde{V} + \rho^2\,\mathrm{d}\theta^2 + \\ &+ \rho^{-2}[(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta] \sin^2\theta\, \mathrm{d}\tilde{\phi}^2 - \\ &- 2a \sin^2\theta\, \mathrm{d}\tilde{\phi}\,\mathrm{d}r - 4a\rho^{-2} Mr \sin^2\theta\, \mathrm{d}\tilde{\phi}\,\mathrm{d}\tilde{V}. \end{aligned} \tag{4.4.2}$$

The general properties of the geometry of a rotating black hole are best seen on a spacetime diagram in Kerr coordinates (Figure 32). Here the time coordinate $\tilde{t}$ is substituted for the coordinates $\tilde{V}$:

$$\tilde{t} = \tilde{V} - r. \tag{4.4.3}$$

We have already employed such diagrams in the Eddington coordinates in Chapter 2. The case we are considering now is essentially different in that the Kerr metric has the axial but not the spherical spatial symmetry.

Since one of the rotational degrees of freedom (the rotation translating a point along the 'meridians' of θ) is not shown in these diagrams, they display information only on one chosen section (e.g., the equatorial plane $\theta = \pi/2$, as we see on Figure 32). The figure plots several world lines of photons that are important for describing the properties of the Kerr geometry. The first thing to remember is that the closer the coordinates are to the horizon, the more they are twisted around the black hole. The world lines of photons falling into the black hole are mapped by straight lines. In Boyer–Lindquist coordinates (a rigid grid; see Sections 4.2 and 4.3), they would appear twisted. Here the coordinate lines are twisted just as the photon trajectories are, so that these trajectories appear as straight with respect to the coordinates lines (in fact, we chose the coordinate lines precisely to have them coincide with the trajectories of the falling photons). At the static limit r_1 [see (4.3.6) and (4.3.7)], the r, θ, $\tilde{\phi}$ = const world line is a null line tangent to the light cone. At $r < r_1$, all photons and particles necessarily participate in the rotational motion around the black hole, moving at $\mathrm{d}\tilde{\phi}/\mathrm{d}\tilde{t} > 0$. But they can escape from below the static limit to $r > r_1$.

At the horizon, all time-like and null world lines point into the black hole, except a single null line, unique for each point of the horizon, of an 'escaping' photon; this null line is tangent to the horizon. This family of world lines 'winds up' on the horizon (see Figure 32), always staying on it. In Kerr coordinates, the equation of these null geodesics is

$$r = r_+, \quad \theta = \text{const}, \quad \phi = \frac{a\tilde{V}}{r_+^2 + a^2}. \tag{4.4.4}$$

All other photons and particles have to continue falling into the black hole after they reach the horizon.

The Kerr metric is invariant under the transformation $t \to -t$, $\phi \to -\phi$ which transforms incoming light rays into outgoing ones; hence, this transformation can be performed in (4.4.1). If we also substitute $\tilde{V} \to -\tilde{U}$, $\tilde{\phi} \to -\tilde{\phi}_{+}$, then equations $\tilde{U}$ = const, $\tilde{\phi}_{+}$ = const describe the family of outgoing light rays and the coordinate $\tilde{U}$ at infinity coincides with the ordinary coordinate of retarded time. The Kerr metric is obtained in these coordinates from (4.4.2) by the transformation $\tilde{V} = -\tilde{U}$, $\tilde{\phi} = -\tilde{\phi}_{+}$.

In contrast to the Schwarzschild metric, here we do not consider the continuation of the Kerr metric into the region within the horizon. [9] The reason is as follows. In the collapse of a spherical body (generating a Schwarzschild black hole), the spacetime metric beyond the collapsing body is exactly Schwarzschild both inside and outside the black hole. In the collapse of a nonrotating slightly nonspherical body, the metric outside of the black hole rapidly tends to the Schwarzschild metric as $t \to \infty$. We will see in Chapter 12 that the same property holds inside the Schwarzschild black hole. The inner region of the Schwarzschild metric thus describes the real 'insides' of a nonrotating black hole.

These arguments do not hold for the Kerr metric. First, when an arbitrary rotating body contracts and turns into a black hole, the metric outside the body cannot become stationary immediately (and, hence, cannot be a Kerr metric) because gravitational waves are emitted in the course of the collapse. This statement holds both for the region outside the horizon and for that inside it. Outside the horizon all derivations from the Kerr metric are radiated away via gravitational waves, as we will see in Chapter 6, and the limiting metric at $t \to \infty$ is the Kerr solution. In the external spacetime, therefore, the Kerr metric describes the real rotating black hole.

In the region inside the horizon, however, the metric does not tend to the Kerr solution either immediately after the collapse or at later stages. For this reason this solution does not describe (inside the horizon) the inner structure of real rotating black hole (the detailed structure of black hole region inside the horizon is treated in Chapter 12). Note that all the above discussed properties of the black hole spacetime are valid only if $M \geq |a|$. Otherwise the horizon vanishes from the solution, and it ceases to describe the black hole. Pathological features appear [Hawking and Ellis (1973)] so that this solution may hardly relate to reality. From the physics standpoint, the formation of an object with $M < |a|$ requires the compression of a rotating body with such a high angular momentum that at $r \approx r_{+}$ the linear velocity of rotation inevitably exceeds the speed of light. Hereafter we always assume (for noncharged black hole) that $M \geq |a|$.

4.5. Celestial Mechanics Near a Rotating Black Hole

Consider the motion of test particles along geodesics in the gravitational field of a rotating black hole. In the general case, the trajectories are fairly complicated because the field has no spherical symmetry. For detailed analysis, see Bardeen *et al.* (1972), Stewart and Walker (1973), Ruffini and Wheeler (1971), Misner, Thorne and Wheeler (1973), Shapiro and Teukolsky (1983), and Dymnikova (1986). Important

aspects of the gravitational capture of particles by a rotating black hole were treated by Dymnikova (1982) and Bicak and Stuchlik (1976). The references given above cite numerous original publications.

We consider the motion of test particles with respect to the 'absolute' space introduced in Section 4.2, that is, with respect to the rigid lattice of the chronometric reference frame described by the Boyer–Lindquist coordinates (see Section 4.3).

First integrals of motion are written in the form

$$\rho^2\frac{dr}{d\lambda} = \{[E(r^2+a^2)-L_z a]^2 - \Delta[m^2 r^2 + (L_z - aE)^2 + Q^*]\}^{1/2}, \tag{4.5.1}$$

$$\rho^2\frac{d\theta}{d\lambda} = \left\{ Q^* - \cos^2\theta \left[a^2(m^2-E^2) + \frac{L_z^2}{\sin^2\theta} \right] \right\}^{1/2}, \tag{4.5.2}$$

$$\rho^2\frac{d\phi}{d\lambda} = -\left(aE - \frac{L_z}{\sin^2\theta} \right) + \frac{a}{\Delta}[E(r^2+a^2)-L_z a], \tag{4.5.3}$$

$$\rho^2\frac{dt}{d\lambda} = -a(aE\sin^2\theta - L_z) + \frac{r^2+a^2}{\Delta}[E(r^2+a^2)-L_z a]. \tag{4.5.4}$$

Here m is the test particle mass, $\lambda = \tau/m$, where τ is the proper time of the particle, E, is the constant energy of the test particle, L_z is the constant projection of the angular momentum of a particle on the rotation axis of the black hole, and Q^* is the integral of motion found by Carter (1968a): [10]

$$Q^* = p_\theta^2 + \cos^2\theta[a^2(m^2-E^2) + \sin^{-2}\theta\, L_z^2], \tag{4.5.5}$$

where p_θ is the θ component of the four-momentum of the test particle. The motion of an ultrarelativistic particle corresponds to the limit as $m \to 0$.

First we consider the characteristic features of the motion of particles in the equatorial plane of a rotating black hole. In this case the expressions for $dr/d\lambda$ and $d\phi/d\lambda$ can be written in the form [Shapiro and Teukolsky (1983)]

$$r^3\left(\frac{dr}{d\lambda}\right)^2 = E^2(r^3 + a^2 r + 2Ma^2) - 4aMEL_z - (r-2M)L_z^2 - m^2 r\Delta, \tag{4.5.6}$$

$$\frac{d\phi}{d\lambda} = \frac{(r-2M)L_z + 2aME}{r\Delta}. \tag{4.5.7}$$

These expressions are analogues of Equations (2.8.1)–(2.8.2) for a Schwarzschild black hole. An analysis of the peculiarities of motion is performed in the same way as in Section 2.8. Thus, equating the right-hand side of equation (4.5.6) to zero and solving it for E, we obtain the 'effective' potential. The extremums of the effective potential correspond to circular motion. In this case the expressions for E_{circ} and L_{circ} have the form (for $m = 1$)

$$E_{\text{circ}} = \frac{r^2 - 2Mr \pm a\sqrt{Mr}}{r(r^2 - 3Mr \pm 2a\sqrt{Mr}\,)^{1/2}}, \tag{4.5.8}$$

$$L_{\text{circ}} = \pm \frac{\sqrt{Mr}\,(r^2 \mp 2a\sqrt{Mr} + a^2)}{r(r^2 - 3Mr \pm 2a\sqrt{Mr}\,)^{1/2}}. \tag{4.5.9}$$

The upper signs in these and subsequent formulas correspond to the same direction of rotation of the particle as that of the black hole, and the lower signs to the opposite sense of rotation, so that we always assume that $a \geq 0$.

The radius of the circular, closest to the black hole, orbit (the motion along it is at the speed of light) is

$$r_{\text{photon}} = 2M\left\{1 + \cos\left[\frac{2}{3}\arccos\left(\mp\frac{a}{M}\right)\right]\right\}. \tag{4.5.10}$$

This orbit is unstable.

The unstable circular orbit on which $E_{\text{circ}} = m$ is given by the expression

$$r_{\text{bind}} = 2M \mp a + 2M^{1/2}(M \mp a)^{1/2}. \tag{4.5.11}$$

These values of the radius are the minima of periastra of all parabolic orbits. If the orbit of a particle, which comes in the equatorial plane from infinity where its velocity $v_\infty \ll c$, passes by the black hole closer than r_{bind}, the particle is captured. The radius r_0 of the periastron of the parabolic orbit is determined by the parameter $\tilde{L}$ of the particle:

$$r_0 = M[\tilde{L}^2 + \sqrt{\tilde{L}^4 - (2\tilde{L} - a/M)^2}\,], \tag{4.5.11a}$$

where $|\tilde{L}| < 1 + \sqrt{1 \mp a/M}$. Finally, the radius of the boundary circle separating stable circular orbits from unstable ones is given by the expression

$$r_{\text{bound}} = M\{3 + Z_2 \mp [(3 - Z_1)(3 + Z_1 + 2Z_2)]^{1/2}\}, \tag{4.5.12}$$

where

$$Z_1 = 1 + (1 - a^2/M^2)^{1/3}[(1 + a/M)^{1/3} + (1 - a/M)^{1/3}],$$

$$Z_2 = (3a^2/M^2 + Z_1^2)^{1/2}.$$

Table II

Orbit	$a = 0$	$a = M$	
		$L > 0$	$L < 0$
r_{photon}	1.5	0.5	2
r_{bind}	2	0.5	2.92
r_{bound}	3	0.5	4.5

Table II lists r_{photon}, r_{bind}, and r_{bound} for the black hole rotating at the limiting angular velocity, $a = M$, and gives a comparison with the case of $a = 0$ (in units of $r_g = 2GM/c^2$). Note that as $a \to M$, the invariant distance from a point r to the horizon r_+, equal to

$$\int_{r_+}^{r} \frac{dr'}{\Delta(r')^{1/2}},$$

diverges. As a result, this does not mean that all orbits coincide in this limit and lie at the horizon, although at $L > 0$ the radii r of all three orbits tend to the same limit r_+ [see Bardeen *et al*. (1972)].

Finally, we will give the values of specific energy E/m, specific binding energy $(m-E)/m$, and specific angular momentum $|L|/mM$ of a test particle at the last stable orbit, r_{bound} (see Table III).

Table III

Orbit	$a = 0$	$a = M$	
		$L > 0$	$L < 0$
E/m	$\sqrt{8/9}$	$\sqrt{1/3}$	$\sqrt{25/27}$
$(m-E)/m$	0.0572	0.423	0.0377
$\lvert L\rvert/mM$	$2\sqrt{3}$	$2/\sqrt{3}$	$22/3\sqrt{3}$

Equation (4.5.6) shows that particle motion with negative E is possible in the neighborhood of a rotating black hole. Let us solve this equation for E:

$$E = \frac{2aML + [L^2r^2\Delta + m^2r\Delta + r^3(dr/d\lambda)^2]^{1/2}}{r^3 + a^2r + 2Ma^2}. \tag{4.5.13}$$

The positive sign of the radical was chosen because it corresponds to the direction of the four-momentum of the particle into the future [Misner, Thorne, and Wheeler (1973)]. The numerator in (4.5.13) is negative if $L < 0$ and the first term is of greater magnitude than the square root of the expression in brackets.

The second and third addends in brackets can be made arbitrarily small ($m \to 0$ corresponds to the transition to an ultrarelativistic particle, and $dr/d\lambda \to 0$ is the transition to the motion in the azimuthal direction). Then E may become negative if

we choose points inside the ergosphere, $r < r_l$. Additional constraints appear if $m \neq 0$ and $dr/d\lambda \neq 0$.

The expression (4.5.13) holds only for $\theta = p/2$. It can be shown that orbits with negative E are possible within the ergosphere for any $\theta \neq 0$. Orbits with $E < 0$ make it possible to organize processes that extract the 'rotational energy' of the black hole. Such processes were discovered by Penrose (1969). This phenomenon and its physical implications are discussed in detail in Section 8.1.

Now we consider some forms of motion of test particles off the equatorial plane, primarily of nonrelativistic particles moving at parabolic velocity ($v_\infty = 0$) and zero angular momentum ($L = 0$). Such particles fall at constant θ and are dragged into the rotation around the black hole in the latitudinal direction at angular velocity (4.3.11), that is, at the angular velocity of locally 'nonrotating observers'. Therefore, these particles fall radially at each point in the reference frame of locally 'nonrotating observers'.

Another important case is the falling of ultrarelativistic particles (photons) which move at infinity at $d\theta/d\lambda = 0$ and $L_z = aE \sin^2 \theta$. Equations (4.5.1)–(4.5.4) reduce for such particles to

$$\frac{dr}{d\lambda} = -E, \quad \frac{d\theta}{d\lambda} = 0, \quad \frac{d\phi}{d\lambda} = \frac{aE}{\Delta}, \quad \frac{dt}{d\lambda} = \frac{(r^2 + a^2)E}{\Delta}. \tag{4.5.14}$$

The world lines of these photons were used in constructing the Kerr coordinate system (Section 4.4).

4.6. Gravitational Capture of Particles by a Rotating Black Hole

By analogy to Section 2.9, consider the gravitational capture of particles by a rotating black hole [this topic is reviewed in: Dymnikova (1986)].

The impact parameter $b_\perp$ of capturing a nonrelativistic particle moving in the equatorial plane is given by the expression

$$b_\perp = \pm\, 2M \frac{1}{v_\infty} \left(1 + \sqrt{1 \mp \frac{a}{M}}\right). \tag{4.6.1}$$

The capture cross-section for particles falling perpendicularly to the rotation axis of the black hole with $a = M$ is plotted in Figure 33 [Young (1976)]. In this case the cross-section area is

$$\sigma_\perp = 14.2\pi (1/v_\infty)^2 M^2. \tag{4.6.2}$$

The impact parameter of particles falling parallel to the rotation axis, $b_\parallel$, can be found in the following manner. Let us denote $\tilde{b}_\parallel = b_\parallel/M$, $\tilde{a} = a/M$. Then $\tilde{b}_\parallel$ is found as the solution of the equation

$$(1 - \tilde{a})^2 q_0^4 + 4(5\tilde{a}^2 - 4)q_0^3 - 8\tilde{a}^2(6 + \tilde{a}^2)q_0^2 - 48\tilde{a}^4 q_0 - 16\tilde{a}^6 = 0, \tag{4.6.3}$$

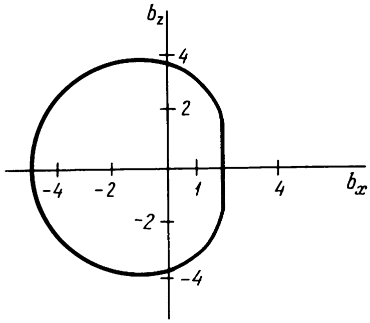


Fig. 33. The capture cross-section for particles moving with $v_\infty \to 0$ at right angles to the rotation axis of a black hole with $a=M$ Coordinate axes are marked off in units of $M(1/v_\infty)$.

where $q_0 = v_\infty^2(\tilde{b}_\parallel^2 - \tilde{a}^2)$. If $\tilde{a} = 1$, then

$$\tilde{b}_\parallel = 3{,}85\left(\frac{1}{v_\infty}\right)M, \quad \sigma_\parallel = 14.8\pi\left(\frac{1}{v_\infty}\right)^2 M^2. \tag{4.6.4}$$

Consider now ultrarelativistic particles. The impact parameters of capture, $b_\perp$, for the motion in the equatorial plane are given by the following formula:

If the angular momentum is positive, then

$$\frac{b_\perp^+}{M} = 8\cos^3\left[\frac{1}{3}(\pi - \arccos\tilde{a})\right] + \tilde{a}. \tag{4.6.5}$$

If the angular momentum is negative, then

$$\frac{b_\perp^-}{M} = -8\cos^3\left(\frac{1}{3}\arccos|\tilde{a}|\right) + \tilde{a}. \tag{4.6.6}$$

In this case the cross-section for $\tilde{a} = 1$ is

$$\sigma_\perp = 24.3\pi M^2. \tag{4.6.7}$$

For photons propagating parallel to the rotation axis of the black hole with $\tilde{a} = 1$, we have

$$\frac{b_\parallel}{M} = 2(1 + \sqrt{2}), \quad \sigma_\parallel = 23.3\,\pi M^2. \tag{4.6.8}$$

A comparison of cross-sections given in this section with those of Sections 2.9 demonstrates that a rotating black hole captures incident particles with lower efficiency that a nonrotating black hole of the same mass does.

4.7. Wave Fields Around a Rotating Black Hole

Weak wave fields (those not perturbing the background metric) in the Kerr metric are studied in the same way that this problem was treated in the case of Schwarzschild metric. The main difficulty arises because the metric has only the axial symmetry (not spherical, as in the Schwarzschild case). Carter, who found the fourth integral and separated the variables in the equations of geodesics in the Kerr metric (see Section 4.5), was able to demonstrate that the equation of a massive (charged) spin-0 field in this metric also admits the separation of variables [Carter (1968b)]. Teukolsky (1972) successfully decoupled the system of equations for the components of the spin-1 and -2 massless fields and reduced these equations to a single generating wave equation for a complex scalar function. A similar result was obtained by Teukolsky (1973) for massless spin-$\frac{1}{2}$ field [see also Unruh (1973)]. In Boyer–Lindquist coordinates, (4.2.1)–(4.2.2), this generating equation takes the form

$$\left[\left(\frac{(r^2+a^2)^2}{\Delta}-a^2\sin^2\theta\right)\frac{\partial^2}{\partial t^2}+\frac{4Mar}{\Delta}\frac{\partial^2}{\partial t\,\partial\phi}-\right.$$

$$-\left(\frac{a^2}{\Delta}-\frac{1}{\sin^2\theta}\right)\frac{\partial^2}{\partial\phi^2}-\Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial}{\partial r}\right)-$$

$$-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)-2s\left(\frac{a(r-M)}{\Delta}+\frac{i\cos\theta}{\sin^2\theta}\right)\frac{\partial}{\partial\phi}-$$

$$\left.-2s\left(\frac{M(r^2-a^2)}{\Delta}-r-ia\cos\theta\right)\frac{\partial}{\partial t}+s^2\,\mathrm{cotan}^2\,\theta-s\right]\Psi_{(s)}=0.$$

Here s is the spin of the field. The variables in this equation can be separated and the solution $\Psi_{(s)}$ can be written in the form

$$\Psi_{(s)}=e^{i\omega t+im\phi}\Phi(r)S(\theta).$$

The spheroidal wave functions $S(\theta)$ were analyzed in detail by Fackerell and Crossman (1977). As in the case of the Schwarzschild metric, the solution $\Psi_{(s)}$ of the generating equation makes it possible to find all components of the massless field of interest.

The method of separation of variables was used to analyze the stability of the Kerr metric [Press and Teukolsky (1973), Stewart (1975)] and to study the scattering of electromagnetic, gravitational, and neutrino fields by Kerr black holes [Starobinsky and Churilov (1973), Teukolsky and Press (1974), Chandrasekhar and Detweiler (1975b, 1976), Detweiler (1977), Chandrasekhar (1979b)].

Attempts to separate variables in the Dirac equation with nonzero mass were unsuccessful until Chandrasekhar (1976) suggested a new method in which the separation of variables was carried out before the system of equations was decoupled. Page (1976c) and Toop (1976) extended this approach to the case of the Dirac equation for massive charged particles. Lee (1976), Chitre (1976), Fackerell and Crossman (1976) treated the interacting electromagnetic and gravitational perturbations in the Kerr–Newman metric. A complete exposition of the mathematical theory of propagation of physical fields in the spacetime of rotating black hole can be found in Chandrasekhar's monograph (1983) where references to original publications are also given. [See also Gal'tsov (1986)].

Here we restrict the presentation to a brief description of the main ideas, methods, and physical conclusions. As in the case of a spherical black hole, we focus attention on the propagation of gravitational waves, and also on superradiance which is specific to the rotating black hole.

We begin with the propagation of gravitational waves.

Chandrasekhar and Detweiler (1975b, 1976) and Detweiler (1977) showed that the gravitational perturbations around a rotating black hole with mass M and parameter a can be found by solving the following scalar equation:

$$\frac{d^2\Phi}{dr_*^2} - V\,(r, M, a, l, m, \omega)\Phi = 0. \tag{4.7.1}$$

Here r_* is the generalized 'tortoise' coordinate introduced by Wheeler (1955):

$$dr_* = \Delta^{-1}(r^2 + a^2)\,dr. \tag{4.7.2}$$

If $r \to \infty$, then $r_* \to \infty$, and if $r = r_+$, then $r_* \to -\infty$. The potential barrier V depends, in addition to r, M, and a, on the spheroidal harmonic indices l and m and on the frequency ω. The specific form of this dependence was given in the papers cited above. The dependence of each harmonic on time is given by the function exp $(i\omega t)$. Equation (4.7.1) holds outside the perturbation source. In the presence of a source, the right-hand side of (4.7.1) is nonzero. Specific expressions for the right-hand side of (4.7.1) were given by Detweiler (1977).

With the solution to (4.7.1) known, it is possible to calculate the perturbation components of the metric tensor. The problem thus reduces, as in the case of nonrotating black holes, to an analysis of the solutions of scalar equation (4.7.1). Note that in the case of the Kerr metric the potential barrier V depends, in contrast to that of the Schwarzschild metric, on m and ω.

The asymptotic form of the general solution of Equation (4.7.1) can be written in the following form:

far from the black hole ($r_* \to \infty$)

$$\Phi \sim A_{\text{out}}\, e^{-i\omega r_*} + A_{\text{in}}\, e^{i\omega r_*}; \tag{4.7.3}$$

close to the horizon ($r_* \to -\infty$)

$$\Phi \sim B_{\text{out}}\, e^{-ikr_*} + B_{\text{in}}\, e^{ikr_*} \tag{4.7.4}$$

(here $k = \omega + am/2Mr_+$).

Squared magnitudes of A and B (in general, A and B are complex quantities) are proportional to energy fluxes of the incident and emitted waves at infinity and horizon, respectively.

First we consider the quasinormal oscillation modes of a rotating black hole [Detweiler (1980)].

As in the case of nonrotating black holes, there are those oscillation modes in which there is no wave coming from infinity ($r_* = \infty$) and no wave coming from the horizon ($r_* = -\infty$), that is,

$$A_{\text{in}} = B_{\text{out}} = 0. \tag{4.7.5}$$

Condition (4.7.5) holds only for specific (resonance) frequencies. The resonance frequencies of quasinormal modes were found by Detweiler (1980). Resonance frequencies as functions of the parameter a are plotted in Figure 34. Recall that the time dependence of each harmonic is given by the function exp ($i\omega t$). If $a \neq 0$, the frequencies become functions of m.

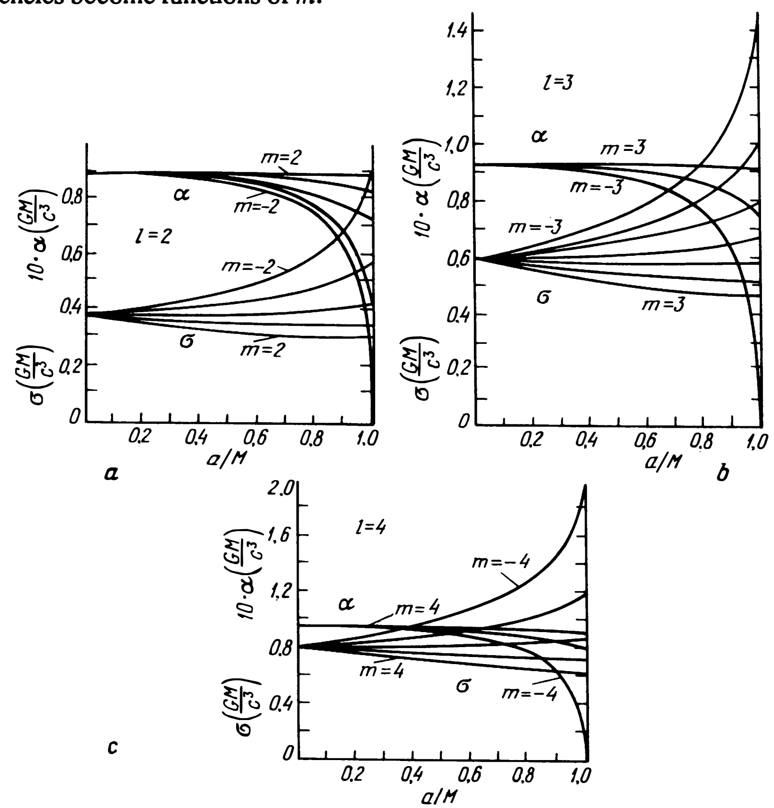


Fig. 34. Resonance frequencies $\omega = \sigma + i\,\alpha$ as functions of the parameter a for different l and m.

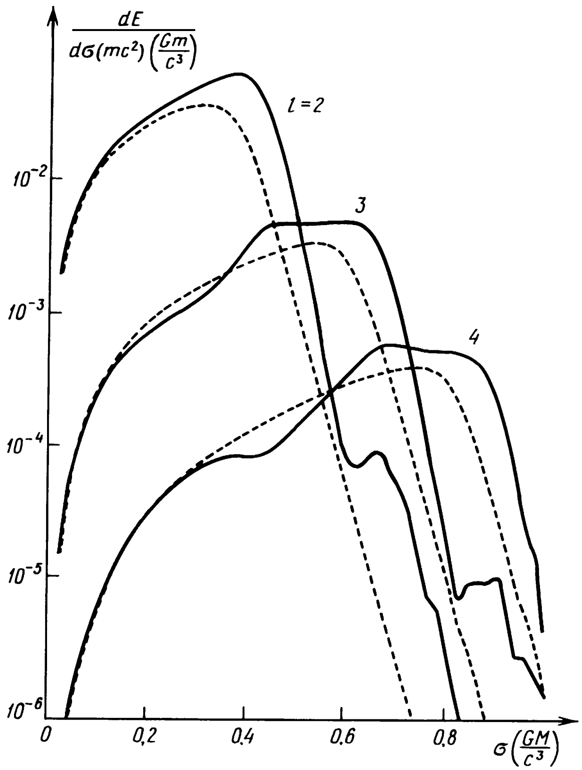


Fig. 35. Total spectrum of gravitational radiation in different modes l by a particle of mass m and $v_\infty=0$. The particle falls along the rotation axis into a black hole with $a/M=0.99$ (solid lines and $a=0$ (dashed lines).

Now let us turn to radiation of gravitational waves by a particle moving in the field of rotating black hole. We begin with a particle falling along the symmetry axis z at a velocity which equals zero at infinity ($v_\infty = 0$) [see Sasaki and Nakamura (1981), Nakamura and Sasaki (1981), Nakamura and Haugan (1983)].

Figure 35 plots the spectrum of radiation emitted by a particle of mass m falling into a black hole with $a/M = 0.99$, for different modes $l = 2,3,4$. Dashed curves shows, for comparison, the spectrum of radiation in the case of radial falling of the particle into the Schwarzschild black hole ($a = 0$).

Two features must be emphasized. First, the predominant part of the emitted energy goes into the 'ringing' radiation of quasinormal modes. Second, most resonance frequencies are higher at $a/M = 0.99$ that at $a = 0$, and hence, the maxima of radiation are also shifted to higher frequencies.

Figure 36 plots the energy emitted in a given mode l as a function of the parameter a of the black hole. Finally, Figure 37 plots the total emitted energy ΔE (a) and the momentum ΔP (b) as functions of a. Recall that ΔP is emitted because the falling particle emits gravitational waves on the way to black hole mostly in the direction of motion.

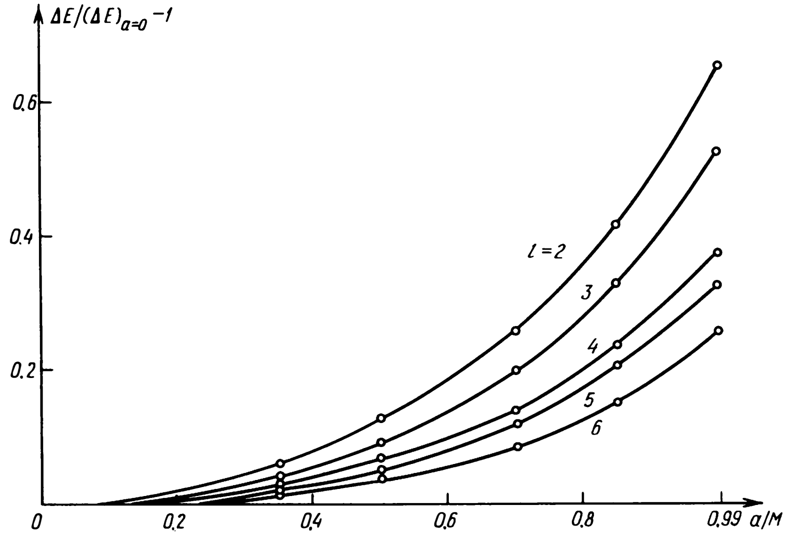


Fig. 36. The ratio of total radiated energy of a particle with v_∞, = 0, falling into a black hole with a given parameter a along the rotation axis, to the energy emitted in the fall into a nonrotating black hole of the same mass, for different modes l.

Now we switch to the case of motion in the equatorial plane of the rotating black hole [Kojima and Nakamura (1983, 1984a,b)]. Figure 38 shows that the total amount of emitted energy for particles with moderate angular momentum which are ultimately captured by the black hole (for a number of values of a).[11] When $a/M \neq 0$, the curves for positive and negative $\tilde{L}$ are apparently asymmetric. First of all, the minima on the curves for $a/M \geq 0.7$ lie at $\tilde{L} \approx -1$ (not at $\tilde{L} = 0$ as in the case $a = 0$). The reason for this is that the tangential components of velocity of particles with negative $\tilde{L}$ diminishes in the neighborhood of the black hole, so as to drag the orbital motion in the direction of the rotation of the black hole. As a result,

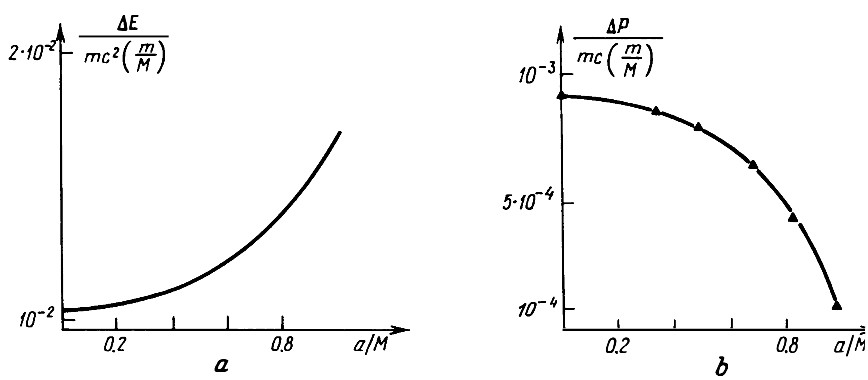


Fig. 37. Total energy (a) and momentum (b) radiated off by a particle of mass m and v_∞=0, falling along the rotation axis, as a function of parameter a.

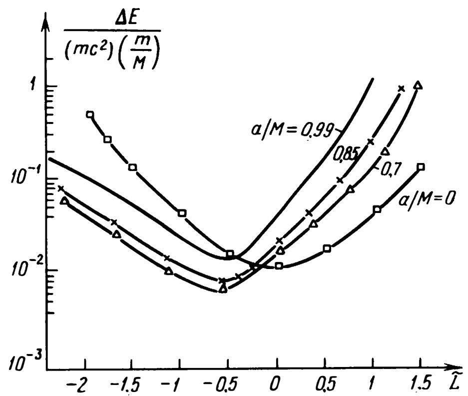


Fig. 38. Total energy radiated by a particle with $v_\infty=0$ falling into a black hole along its equatorial plane, as a function of angular momentum $\tilde{L}$ of the particle, for different values of the black hole parameter a.

particles with negative $\tilde{L}$ of low magnitude have smaller velocity than those with, for example, $\tilde{L} = 0$, and correspondingly radiate away a smaller amount of energy. The radiation of particles with negative $\tilde{L}$ of large magnitude is mostly determined by the magnitude of the angular momentum; hence, ΔE increases as $|\tilde{L}|$ increases.

Because of the asymmetry of the ΔE vs $\tilde{L}$ curves, more energy is emitted at $\tilde{L} > 0$, for the same magnitude of $\tilde{L}$, that at $\tilde{L} > 0$. According to Kojima and Nakamura, the following factor is responsible for the main part of this asymmetry The capture of a particle by the black hole (as discussed above) excites the

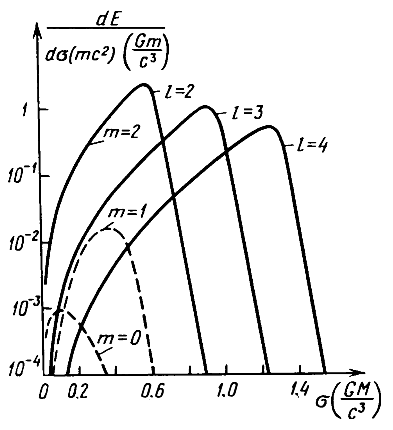


Fig. 39. Total radiation spectrum in different modes, for a particle with $\tilde{L} = 1.3$ falling into a black hole with $a/M = 0.85$ in its equatorial plane.

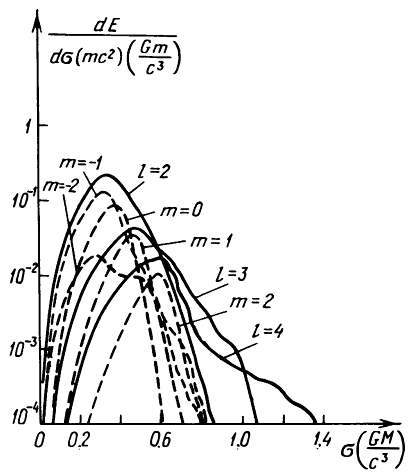


Fig. 40. Same as in Figure 39 but for $\tilde{L} = -2.25$.

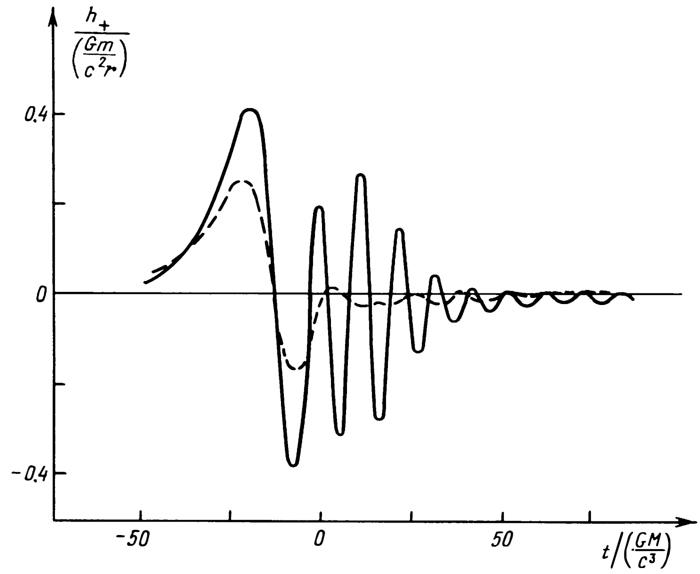


Fig. 41. Gravitational waveform h_+ for $l = 2$ in the equatorial plane and for $\phi = 0$, under the same conditions as in Figure 39 (solid curve) and in Figure 40 (dashed curve).

oscillation of its eigenmodes. If the particle moves in the direction of rotation of the black hole, the mode with $m = -l$ (for fixed l) [12] is excited with maximum probability. According to Figure 34, however, the imaginary part of ω is the smallest precisely for $m = -l$. Hence, this mode undergoes the lowest damping and ultimately its contribution to radiation is predominant (that is clearly seen in Figure 39 where the contributions of different m are shown for $l = 2$).

The least-damped mode is thus excited stronger than others.

If a particle has a negative $\tilde{L}$, the excitation is the strongest for modes with $m = l$ (for fixed l). But Figure 34 shows that these modes undergo maximum damping. As a result, the contributions of different m become comparable (Figure 40) and the total amount of the emitted energy is appreciably lower than in the preceding case. The difference in the radiation of a black hole capturing a particle moving in the direction of rotation and a particle moving in the opposite direction is also clearly seen in Figure 41 which shows the time evolution of the perturbation of the gravitational field in the wave. The 'ringing' radiation lasts longer and the wavelength is shorter in the former case (the magnitude of $\tilde{L}$ are approximately the same).

Finally, Figure 42 shows the radiation of the angular momentum ΔL and momentum ΔP as a function of $\tilde{L}$ for black holes with different values of a.

Let us now consider the emission of gravitational waves by a particle moving in the plane $\theta = \pi/2$. We assume that its velocity at infinity is zero and the angular momentum $\tilde{L}$ is sufficiently large for not being captured by the black hole (the case of scattering). This problem was treated by Kojima and Nakamura (1984b).

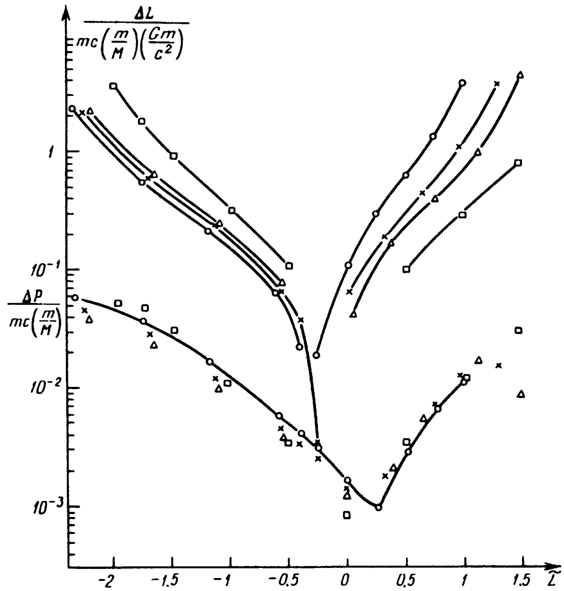


Fig. 42. Total radiated angular momentum and momentum for the conditions of Figure 38: circles - a/M= 0.99, crosses - 0.85, triangles - 0.7, squares - 0.

Recall that the scattering of a nonrelativistic particle by a nonrotating black hole did not excite the oscillation eigenmodes of the black hole (see Section 3.3). The periastron of the trajectory of such a particle lies beyond the potential barrier for gravitational waves, and the waves emitted by the moving particle have frequencies that are too low for the penetration through the barrier and for exciting the oscillation eigenmodes of the black hole. However, if the black hole is rotating, the frequency of gravitational waves emitted at the periastron (it is equal to twice the angular velocity of motion of the particle) may be quite close to the eigenfrequency

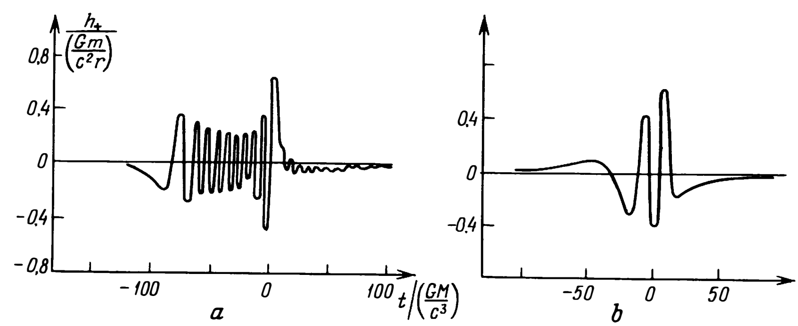


Fig. 43. Gravitational waveform h_+ for $l = 2$, $m = -2$ in the equatorial plane $\theta = \pi/2$ and along the direction to the periastron of the trajectory of scattered particle, $\phi = 0$ and v_∞=0, in the field of a black hole with $a/M = 0.99$: (a) $\tilde{L} = 2.21$, (b) $\tilde{L} = 2.6$.

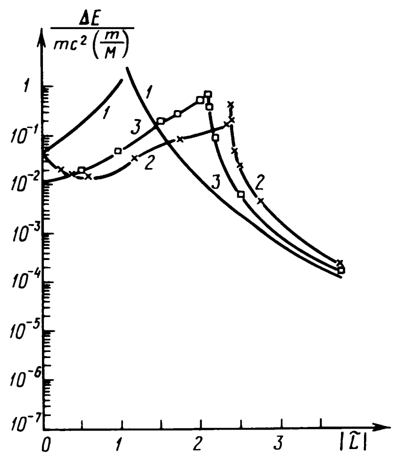


Fig. 44. Total energy radiated by a particle with $v_\infty = 0$ falling in the equatorial plane of a black hole, as a function of $|\tilde{L}|$: 1 - $\tilde{L} > 0$, $a/M = 0.99$, 2 - $\tilde{L} < 0$, $a/M = 0.99$, 3 - $a = 0$.

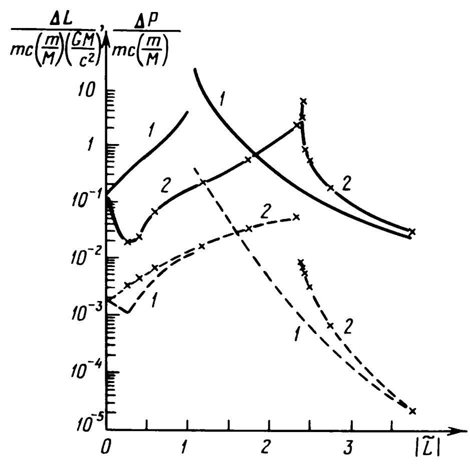


Fig. 45. Total radiated angular momentum (solid line) and momentum (dashed line). The same notation as in Figure 44.

and sufficient for its excitation. This can occur if $\tilde{L}$ is positive and only slightly exceeds the capture threshold.

Thus, Figures 43a and 43b show the perturbation of the metric in the case of the motion around a black hole with $a/M = 0.99$ and $\tilde{L} = 2.21$ and 2.6. No 'ringing' radiation is seen in the second case. If the first case, an undamped train of waves due to the prolonged rotation of the particle close to the critical capture circle is seen, followed by a damped 'ringing' radiation. A simple estimate shows that in the first case the frequency of gravitational waves emitted at the periastron in nearly equal to the eigenfrequency of the $l = -m = 2$ mode.

The emitted energy, angular momentum, and momentum as functions of $\tilde{L}$ are plotted in Figures 44 and 45 for a black hole with $a/M = 0.99$.

An extremely interesting phenomenon called superradiance will be treated in detail in Section 8.1. It is logical to take it up after the general theory of black holes has been presented (Section 6). On the other hand, though, the phenomenon of superradiance belongs to the propagation of waves in the neighborhood of rotating black holes. For this reason, here we briefly expose the physical meaning of the phenomenon and give its quantitative characteristics.

Let us consider the process of irradiation of a rotating black hole by a wave (say, gravitational wave) of a fixed frequency σ and fixed multipolarity. As has already been shown for the Schwarzschild black hole, the incident wave can (in the general case) partly penetrate through the potential barrier and be absorbed by the black hole, and partly scatter to infinity (Section 3.2). The amplitude of the scattered waves is always less than (or equal to) the amplitude of the incident wave, since a part of the incident energy is absorbed by the black hole. An analysis of the situation in the case of a rotating hole reveals that the amplitude of the scattered

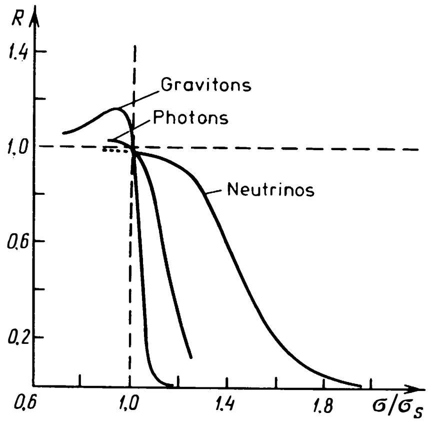


Fig. 46. Reflection coefficient R as a function of σ.

wave can be enhanced, for certain parameters of the incident wave, in comparison with the incident amplitude. This is what the term *superradiance* means [Zel'dovich (1970)]. The additional energy is drawn from the 'rotational energy' of the black hole. It is shown in Section 8.1 that superradiance can arise if the frequency of the wave lies in the interval

$$0 < \sigma < \sigma_S, \tag{4.7.6}$$

where

$$\sigma_S = -\frac{am}{2Mr_*} \tag{4.7.7}$$

(m is negative).

The calculation of superradiance reduces to analyzing the properties of the solution of equations of type (4.7.1) [see, e.g., Chandrasekhar (1979b)]. Figure 46 reproduced from the cited paper shows the plots of the reflection coefficient R as a function of a frequency of the incident waves for different types of waves (R is the ratio of squared amplitude of the reflected to incident waves). The figure reveals superradiance, that is, $R > 1$, for gravitational and electromagnetic waves in the range $\sigma < \sigma_S$. No superradiance is found for neutrinos. The reasons for this were analyzed by Martellini and Treves (1977), Chandrasekhar (1979b), Iyer and Kumar (1979).

Note also that beyond the event horizon the Kerr metric is apparently stable, as the Schwarzschild metric is, under small perturbations [for the review of the problem, see Thorne (1976)]. On the instability of the Kerr metric inside the event horizon, see Section 12.4.

To conclude the section, consider the scattering of a parallel beam of waves that are incident on a rotating black hole. By the time this monograph has been completed, only the data on the scattering of a beam of gravitational waves

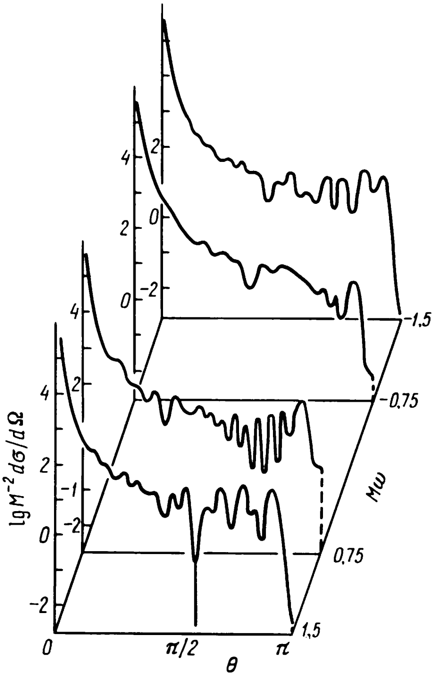


Fig. 47. The differential cross-section of the scattering of gravitation radiation by a rotating black hole with $a/M = 0.9$, for different frequencies ω.

propagating parallel to the rotation axis of the black hole was published [Handler and Matzner (1980)]. This problem is similar to that of wave scattering by a Schwarzschild black hole (see Section 3.5) and can be solved by similar methods. The scattering under the geometrical optic approximation was discussed in the preceding section.

Figure 47 [Handler and Matzner (1980)] plots differential cross-sections of scattering for gravitational waves incident parallel to the rotation axis of a black hole with $a/M = 0.9$. Here ω denotes frequency and σ denotes cross-section. Positive ω corresponds to the circular polarization of waves in the direction of rotation of the black hole, and negative ω , to the opposite circular polarization. The figure shows that the main features of the differential cross-section are similar to those in the case of nonrotating black hole, although rotation introduced considerable complications. The total cross-sections of absorbing the radiation, σ , are

M_ω	0.75	1.5	−0.75	−1.5
σM^{-2}	36.5	62.5	88.7	80.3

Finally, Figure 48 plots the differential cross-section for a black hole with $a/M = 0.99$ and $M\omega = 0.75$. Here the effect of superradiance is very important. The total absorption cross-section is negative and equal to $-15.8\, M^2$. A comparison of this

case with that analyzed above ($a/M = 0.9, M\omega = 0.75$) shows that superradiance is

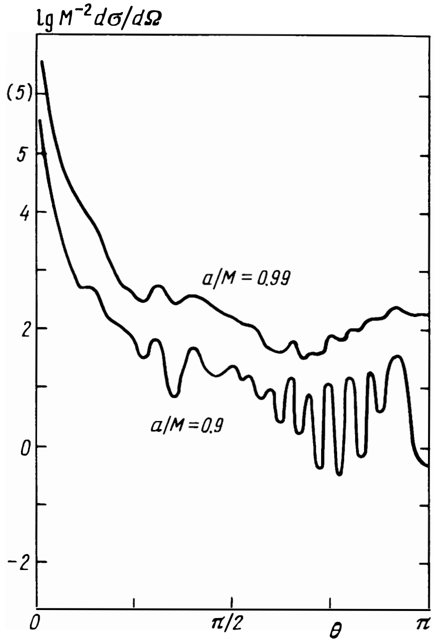


Fig. 48. Two differential scattering cross-sections for gravitational radiation with $M\omega = 0.75$ by a black hole with $a/M = 0.9$ and $a/M = 0.99$. Superradiance is an important effect in the last case. The curve representing this situation is shifted upwards by unity along the ordinate axis (to make the plot clearer). The number 5 in parentheses on the ordinate axis refers only to this curve.

added to ordinary scattering, 'filling up' the minima on the curves. This effect is especially well pronounced in backscattering ($\theta = \pi$).

4.8. Charged Rotating Black Hole

In any realistic situation, the electric charge of a black hole is negligible. As a rule, the ratio of the charge Q to mass M of a black hole cannot exceed 10^{-18} [Wald (1984)]. Indeed, as the charge-to-mass ratio of the electron and the proton is $(q/m)_e = 10^{21}$ and $(q/m)_p = 10^{18}$, respectively, and the ratio of the gravitational force of the electrostatic one in the interaction of these particles with the black hole of charge Q and mass M is, on the order of magnitude, qQ/mM, the ratio Q/M cannot be greater than $(q/m)^{-1}$, otherwise charges of like sign would be repelled from the black hole while charges of the opposite sign would fall into it and neutralize the electric charge of the black hole.

On theoretical grounds, however, it would be interesting to discuss, at least briefly, the general case of a rotating charged black hole.

In this case the metric of spacetime is written in the form of (4.2.1) but now the expression for Δ is a function of charge Q (the Kerr–Newman geometry):

$$\Delta = r^2 - 2Mr + a^2 + Q^2. \tag{4.8.1}$$

In addition to the gravitational field, the black hole is now surrounded with a stationary electromagnetic field which is completely determined by the charge Q and parameter a. The vector potential of this field in coordinates (4.2.1), (4.8.1) is written in the form

$$A_\alpha \, dx^\alpha = -\frac{Qr}{\rho^2}(dt - a \sin^2\theta \, d\phi). \tag{4.8.2}$$

If $a = 0$, the black hole does not rotate and the metric represents a spherically symmetric charged black hole with spherically symmetric electric field [the Reissner (1916)–Nordström (1918) solution].

If the black hole rotates ($a \neq 0$), the electric field is supplemented with a magnetic field due to the dragging of the inertial reference frame into the rotational motion around the black hole.

At large distances from a black hole in the 'rigid' reference frame (chronometric frame; see Section 4.3), which transforms at infinity into the Lorentz frame, the largest components of the electromagnetic field correspond to a monopole electric field with charge Q and a dipole magnetic field with magnetic moment $\mu^* = Qa$. Other moments of the field are also expressed in terms of Q and a in a unique manner [for details, see Cohen and Wald (1971)], Hanni and Ruffini (1973)]. If the electric charge is nonzero, the Kerr–Newman solution contains a horizon, provided $M^2 \geq Q^2 + a^2$, that is, the solution describes a black hole only if this condition is satisfied; we consider only such solutions (cf. the discussion in Section 4.4).

The motion of a test particle in the Kerr–Newman metric can be written in the form similar to (4.5.1)–(4.5.4). We denote by E the conserved energy of the particle with charge q and mass m, and by L_z the conserved projection of angular momentum on the black hole axis: we have

$$E = -(p_t + qA_t), \quad L_z = p_\phi + qA_\phi, \tag{4.8.3}$$

where p_α is the four-momentum of the particle. The equations of motion are written in the form

$$\rho^2 \frac{dr}{d\lambda} = \{[E(r^2 + a^2) - L_z a - qQr]^2 - \Delta[m^2 r^2 + (L_z - aE)^2 + Q^*]\}^{1/2}, \tag{4.8.4}$$

$$\rho^2 \frac{d\theta}{d\lambda} = \left\{ Q^* - \cos^2\theta \left[a^2(m^2 - E^2) + \frac{L_z^2}{\sin^2\theta} \right] \right\}^{1/2}, \tag{4.8.5}$$

$$\rho^2\frac{d\phi}{d\lambda} = -\left(aE - \frac{L_z}{\sin^2\theta}\right) + \frac{a}{\Delta}[E(r^2+a^2) - L_z a - qQr], \tag{4.8.6}$$

$$\rho^2\frac{dt}{d\lambda} = -a(aE\sin^2\theta - L_z) + (r^2+a^2)\Delta^{-1}[E(r^2+a^2) - L_z a - qQr] \tag{4.8.7}$$

[the expression for Q^* is given in (4.5.5)].

It must be emphasized that in this general form, the equations describe not only the phenomena specific of black holes (these were mostly discussed in the preceding sections) but also their combination with ordinary effects caused by the motion of a test particle in electromagnetic field.

The physical fields in the Kerr–Newman spacetime share quite a few properties with the above-discussed fields in the Schwarzschild geometry and in a rotating black hole. In addition, a qualitatively new phenomenon arises in the field of a rotating charged black hole: the mutual transformation of electromagnetic and gravitational waves. This phenomenon is discussed in Chapter 8.

Wave propagation in the Reissner–Nordström metric and the proof of the stability of this metric outside the event horizon were treated by Bicak (1972, 1979), Sibgatullin and Alekseev (1974), Moncriff (1974c, 1975), Zerilli (1974), Chandrasekhar and Xanthopoulos (1979), and Sibgatullin (1984). The complete mathematical treatment of this problem for a rotating black hole is presented in the already cited book by Chandrasekhar (1983). On the stability of the Reissner–Nordström metric inside the event horizon, see Sections 12.2 and 12.3.

Chapter 5

General Properties of Black Holes

5.1. Asymptotically Flat Spaces. Penrose Diagrams

So far we have restricted the description of the properties of black holes to an analysis of the Schwarzschild and Kerr metrics. The respective spacetime was stationary and possessed certain symmetry. A study of geodesics (corresponding to the motion of test particles and light rays) and wave fields in these metrics enabled us to describe a number of interesting and important features of physical effects in the field of such black holes. A natural question arises: Are there black holes that differ from those already described? What are their properties? In order to answer these questions, the definition must first of all be extended to the general case of nonstationary spacetime. This generalization is obvious. It is reasonable to define the black hole in general as the region of spacetime from which no information-carrying signals are allowed to escape to a distant observer.

In order to make this definition rigorous, one must elaborate what class of observers is meant and what is understood by 'distant' in geometrically invariant terms. The necessary refinement is easily achieved in the physically important case in which there are no matter and no sources of fields far from the black hole. The greater the distance from the black hole, the smaller the deviations are of the spacetime geometry from flatness. The spacetime possessing this property is said to be asymptotically flat.

The need for rigorous definitions of seemingly clear concepts is obvious in studying black holes, because the very existence of these objects modifies the structure of spacetime and its global properties in a principal way as compared with flat Minkowski spacetime. For example, the Schwarzschild spacetime contains a singularity and some geodesics do not extend to infinity. Note that one such geodesic is, for instance, the circular orbit of light ray at $r = 1.5\, r_g$; this geodesic lies completely outside the black hole. The formation of black holes, their dynamic interaction, and their merging may produce especially complicated situations. Semi-intuitive, visually convincing arguments are clearly inadequate here.

A rigorous definition of asymptotically flat spaces was suggested by Penrose (1963). The following line of reasoning can be used to arrive at this definition.

Let us consider first the structure of flat Minkowski spacetime at infinity. To do this, we use an approach typical in geometry: perform an appropriate conformal

transformation that brings the infinitely removed points to a finite distance. First we transform the ordinary spherical coordinates t, r, θ, ϕ in spacetime M to new coordinates ψ, ξ, θ, ϕ using the following relations:

$$t + r = \tan \frac{1}{2}(\psi + \xi), \quad \theta = \theta, \tag{5.1.1a}$$

$$t - r = \tan \frac{1}{2}(\psi - \xi), \quad \phi = \phi, \tag{5.1.1b}$$

$$-\pi/2 \leq \psi - \xi \leq \psi + \xi \leq \pi/2.$$

Then the interval ds^2 now takes the form

$$ds^2 = \Omega^{-2}\, d\tilde{s}^2, \quad d\tilde{s}^2 = -d\psi^2 + d\xi^2 + \sin^2 \xi\, d\omega^2,$$

$$d\omega^2 = d\theta^2 + \sin^2 \theta\, d\phi^2, \tag{5.1.2a}$$

where

$$\Omega = 2 \cos \frac{1}{2}(\psi + \xi) \cos \frac{1}{2}(\psi - \xi). \tag{5.1.2b}$$

Points at infinity in Minkowski spacetime correspond to $\psi + \xi$ and $\psi - \xi$ assuming the values $\pm\pi/2$. At these values the metric ds^2 becomes meaningless but the metric $d\tilde{s}^2$ conformal to ds^2 is regular. [1] When studying the conformal structure on a manifold with a boundary, (5.1.1), we thereby study the conformal structure of Minkowski spacetime, including the infinitely removed points. Recall that it is the conformal structure that is important for studying the general properties of spacetime because it determines the causal properties of the neighborhood of a point, including the properties of null cones.

The $d\tilde{s}^2$ metric is the metric of the four-dimensional cylinder $S^3 \times R^1$ (Figure 49a). Inequalities (5.1.1b) cut a region corresponding to M on the cylinder; it is hatched in Figure 49a. (Of course, we can show only two of the four coordinates; θ and ϕ are omitted.) For the region M cut out of the cylinder, dissected at the point I^0 and developed on a plane, the result is shown in Figure 49b. This is the form typically chosen to represent the conformal Minkowski world. This is the so-called *Penrose diagram* for M. Remember that the left- and right-hand points I^0 coincide, that is, must be 'glued together'.

All time-like curves in the Penrose diagrams begin at the point I^- and end at the point I^+, and all spatial sections pass through I^0. Therefore I^- is said to be the *past time-like infinity*, I^+ is the *future time-like infinity*, and I^0 is the *spatial infinity*.

All null geodesics in M begin at the boundary $\mathcal{J}^-$ (at the future null cone of the point I^-) and end at $\mathcal{J}^+$ (at the past null cone of I^+). the boundaries $\mathcal{J}^-$ and $\mathcal{J}^+$ are said to be the *past null infinity* and *future null infinity*, respectively. Clearly, M is bounded by the 'infinities' $\mathcal{J}^-$ and $\mathcal{J}^+$ and the points I^-, I^-, I^0.

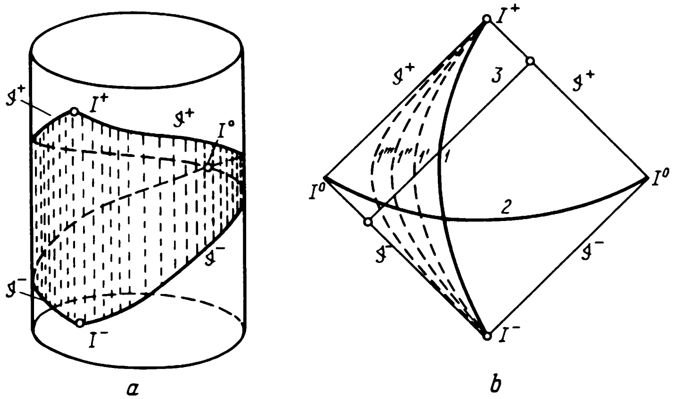


Fig. 49. (a) Conformal structure of Minkowski spacetime. (b) Penrose diagram of Minkowski spacetime. The diagram shows time-like (1, 1', ...), space-like (2), and null (3) geodesics.

Penrose diagrams are also convenient for studying the global structure of spacetime when geometry is essentially nonflat. The coordinates typically used for this purpose represent null rays by straight lines at 45° angles, (e.g., the coordinates ψ and ξ used above possess this property). The causal structure dictated by the arrangement of local null cones is especially clear in such coordinates. Obviously, two-dimensional Penrose diagrams map the geometry of certain two-dimensional sections of spacetime.

Let us now return to the problem of infinitely distant observers. The world lines of such observers which are at rest at the points r, r', r'', r''' ($r < r' < r'' < r'''$) are shown in Figure 49b by the lines 1, 1', 1'', 1''', respectively. The greater r, the closer to $\mathscr{I}^-$ and $\mathscr{I}^+$ the corresponding line is. As $r \to \infty$, it tends to $\mathscr{I}^-$ and $\mathscr{I}^+$. It is thus logical to refer to $\mathscr{I}^-$ and $\mathscr{I}^+$ as infinitely distant boundaries of M. Note that the factor Ω in (5.1.2), which performs conformal transformation vanishes at $\mathscr{I} \equiv \mathscr{I}^- \cup \mathscr{I}^+$ and its gradient $\partial\Omega/\partial x^\mu|_\mathscr{I} \neq 0$ is a null vector tangent to the generators of the surface $\mathscr{I}$.

Sometimes it is more convenient to study the world close to $\mathscr{I}$ using coordinates other than (5.1.1). Note that the retarded null coordinate $u = t - r$ can be used to write the interval in Minkowski world:

$$ds^2 = -du^2 - 2\,du\,dr + r^2\,d\omega^2. \tag{5.1.3}$$

The transformation $\rho = r^{-1}$ then gives the following conformal form of the metric

$$ds^2 = \Omega^{-2}\,d\tilde{s}^2, \quad d\tilde{s}^2 = -\rho^2\,du^2 + 2\,du\,d\rho + d\omega^2,$$

$$\Omega = \rho = r^{-1}. \tag{5.1.4}$$

In these coordinates the surface $\mathscr{I}^+$ is described by the equation $\rho = 0$. A point on $\mathscr{I}^+$ with coordinates u_0, θ_0, ϕ_0 corresponds to the 'end point' $r \to \infty$ of the outgoing

null ray $u = u_0$, $\theta = \theta_0$, $\phi = \phi_0$. Likewise $\mathcal{J}^-$ can be described by replacing u with the advanced null coordinate $v = t + r$.

Assuming that the properties of asymptotic flat spaces in the neighborhood of 'infinity' must be similar to those of Minkowski space, Penrose suggested the following definitions (1963, 1964, 1965, 1968).

The so-called asymptotically simple spacetimes are defined first.

A spacetime M with metric $g_{\mu\nu}$ is said to be *asymptotically simple* if there exists another, 'nonphysical', space $\tilde{M}$ with boundary $\partial\tilde{M} \equiv \mathcal{J}$ and a regular metric $\tilde{g}_{\mu\nu}$ on it,[2] such that (i) $\tilde{M}\backslash\partial\tilde{M}$ is conformal to M, and $g_{\mu\nu} = \Omega^{-2}\tilde{g}_{\mu\nu}$ in M; (ii) $\Omega|_M > 0$, $\Omega|_{\partial\tilde{M}} = 0$, $\Omega_{,\mu}|_{\partial\tilde{M}} \neq 0$; (iii) each null geodesic in M begins and ends on $\partial\tilde{M}$.

Penrose proved that if the metric $g_{\mu\nu}$ satisfies in the neighborhood of $\mathcal{J}$ Einstein's vacuum equations (or Einstein's equations with the energy-momentum tensor that decreases at infinity sufficiently fast) and the natural conditions of causality and spacetime orientability are satisfied, then the asymptotically simple space has the following properties:

(1) The topology of the space M is R^4; its boundary $\mathcal{J}$ is lightlike and consists of two unconnected components $\mathcal{J} = \mathcal{J}^+ \cup \mathcal{J}^-$, each with topology $S^2 \times R^1$.

(2) The generators of the surfaces $\mathcal{J}^\pm$ are the null geodesics in $\tilde{M}$; tangent vectors to these geodesics coincide with $\tilde{g}^{\mu\nu}\Omega_{,\mu}|_{\mathcal{J}}$.

(3) The curvature tensor in the physical space M decreases as we move along the null geodesic to infinity, and the so-called *peeling property* holds. Here we do not focus on this property, as its detailed analysis can be found in the literature [see, e.g., Sachs (1964), Penrose (1968)].

Property 1 signifies that the global structure of the asymptotically flat space is the same as that of Minkowski space. For example, it has a similar causal structure and does not 'allow' black holes. In order to take into consideration the possibility of the existence of localized regions of strong gravitational fields which do not alter the asymptotic properties (as $r \to \infty$) of spacetime, it is sufficient to analyze the class of spaces that can be converted into asymptotically simple spaces by 'cutting out' certain inner regions with singularities of some kind (due to the strong gravitational field) and by subsequent smooth 'patching' of the resultant 'holes'. Such spaces are said to be *weakly asymptotically simple.* To be more rigorous, a space M is said to be weakly asymptotically simple if there exists an asymptotically simple space $\tilde{M}$ such that for an open subset of $\tilde{M}$, $K(\partial\tilde{M} \subset K)$, the space $\tilde{M} \cap K$ is isometric to the subset of M. We say that a weakly asymptotically simple space is *asymptotically flat* if its metric in the neighborhood of $\mathcal{J}$ satisfies Einstein's vacuum equations (or Einstein's equations with energy-momentum tensor that decreases sufficiently fast).

The Schwarzschild and Kerr spacetime, (2.2.1) and (4.2.1), are asymptotically flat. The respective Penrose diagrams are shown in Figures 50c and 67. The metric ds^2 of the space which is conformal to the spacetime of eternal Schwarzschild black hole can be obtained from metric (2.7.12) by changing from the Kruskal coordinates $\tilde{u}$, $\tilde{v}$ to the coordinates

$$\Psi = \frac{1}{2}\arctan\frac{2\tilde{v}}{1-(\tilde{v}^2-\tilde{u}^2)}, \quad \xi = \frac{1}{2}\arctan\frac{2\tilde{u}}{1+(\tilde{v}^2-\tilde{u}^2)},$$

$$-\pi/2 < \Psi+\xi < \pi/2, \quad -\pi/2 < \Psi-\xi < \pi/2,$$

$$-\pi/4 < \Psi < \pi/4,$$

and then omitting the conformal factor.

Property 3 given above signifies that the effects due to curvature are negligible in asymptotically flat spaces in the neighborhood of $\mathcal{I}$, and the spacetime itself near $\mathcal{I}$ is nearly flat. Thus, the ordinary law of energy-momentum conservation holds in this region with a good accuracy and the motion of test particles can be treated as approximately uniform and on a straight line. Correspondingly, the group of asymptotic symmetries can be defined in asymptotically flat spaces. Note in this connection that in Cartesian coordinates, the Poincaré transformation of Minkowski space has the form

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu, \tag{5.1.5}$$

where $\Lambda^\mu{}_\nu$ is the Lorentz transformation matrix and a^μ is the translation vector corresponding to a shift of the origin. Now we introduce in Minkowski space retarded (u, r, θ, φ) or advanced (v, r, θ, φ) coordinates and denote by w either the retarded (u) or the advanced (v) time coordinate. Then transformation (5.1.5) corresponds to the following transformation of the coordinates w, r, θ, φ:

$$w' = w'(w, r, \theta,, \phi) \quad r' = r'(w, r, \theta, \phi),$$

$$\theta' = \theta'(w, r, \theta, \phi), \quad \phi' = \phi'(w, r, \theta, \phi). \tag{5.1.6}$$

As $r \rightarrow \infty$, the functions describing this transformation take on a simple form. Thus, the following transformations correspond in this limit to shifts ($\Lambda^\mu{}_\nu = 0$) in physical spacetime:

$$w' = w + a_0 + a_1 \sin\theta\cos\phi + a_2\sin\theta\sin\phi + \alpha_3\cos\theta,$$

$$\theta' = \theta, \quad \phi' = \phi, \tag{5.1.7}$$

Formulas (5.1.7) describe shifting the surface $\mathcal{I}$ along its generators.

This result can be described more formally in a way that admits a natural generalization to the case of arbitrary asymptotically flat spaces. Let ξ^μ be a Killing vector field corresponding to the symmetry transformation in the physical space M ($\nabla^{(\nu}\xi^{\mu)} = 0$); hence, ξ^μ in the conformal space $\tilde{M}$ obeys the equation

$$\tilde{\nabla}^{(\nu}\xi^{\mu)} - \Omega^{-1}\tilde{\nabla}_\alpha\Omega\xi^\alpha\tilde{g}^{\nu\mu} = 0, \tag{5.1.8}$$

where $\tilde{\nabla}^\mu$ is the covariant derivative in the metric $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$. In the general case, equation (5.1.8) has no nontrivial solutions if the spacetime M does not allow exact

isometric. This is true, for example, for general type asymptotically flat spaces. However, if the analysis is restricted to the neighborhood of $\mathscr{I}$ and we stipulate that Equation (5.1.8) is satisfied only on $\mathscr{I}$, it again has nontrivial solutions. These solutions determine vector fields that generate the transformations of asymptotic symmetric. Note the spectacular fact that the group corresponding to these transformations is independent of the choice of a specific representative of the class of asymptotically flat spaces. This group is known as the Bondi–Matzner–Sachs group (the BMS group). A detailed presentation of its properties and a description of its representations can be found in papers by Sachs (1962), Bondi *et al.* (1962), Penrose (1964), McCarthy (1972a, b, 1973), McCarthy and Crampin (1973), Volovich *et al.* (1978). Here we only briefly outline the main properties of this group which are important for the presentation that follows.

The BMS group is infinite-dimensional; hence, it is much wider than the Poincaré group, which exactly preserves the metric of a flat space. It happens because the BMS group preserves only the asymptotic form of the metric and the gravitational field decreases slowly at infinity. An important property of the BMS group is that it contains a uniquely identifiable normal four-dimensional subgroup of translations. The action of this subgroup on $\mathscr{I}$ in Minkowski space coincides with that of (5.1.7). In the general case, it is possible to introduce in the neighborhood of $\mathscr{I}$, in asymptotically flat spaces, the coordinates in which the transformations of the translation subgroup have the form (5.1.7). Such coordinates are known as *conformal Bondi coordinates* [see Tamburino and Winicour (1966)].

We have described a class of asymptotically flat spaces whose asymptotic behavior is similar to the asymptotic behavior of Minkowski space, and briefly outlined their properties. The concept of asymptotically distant observer, that moves almost by inertia, can be very naturally introduced in this class of spaces. Now we can give a rigorous definition of the black hole. But first we will briefly treat one more aspect connected with the scattering of massless fields in asymptotically flat spaces, as it will be useful in subsequent chapters.

The above definition of asymptotically flat space is especially convenient is dealing with the problem of scattering of massless fields, and in particular, convenient for describing the properties of gravitational radiation. The universal asymptotic behavior ($\propto 1/r$) of these fields in the wave zone allows the use of conformal mapping for transforming the problem of scattering in the physical spacetime into a problem with regular initial data at the past null boundary of $\mathscr{I}^-$ in the Penrose space. It is then found that the regular behavior of conformally transformed field in the neighborhood of $\mathscr{I}$ implies a specific form of decrease of this field in the asymptotic region.

We will illustrate these arguments with an example of a scalar massless conformally invariant field described by the equation

$$(\Box - \frac{1}{6}R)\varphi = 0 \tag{5.1.9}$$

in the asymptotically flat space $(M, g_{\mu\nu})$. Now we use the conformal mapping $g_{\mu\nu} = \Omega^{-2}\tilde{g}_{\mu\nu}$ in order to transform to the conformal Penrose space $(\tilde{M}, \tilde{g}, \Omega)$,

supplementing it with a conformal mapping of the field, $\varphi \to \tilde{\varphi} = \Omega^{-1}\varphi$. We refer to the values of the field $\tilde{\varphi}$ at past null infinity $\mathscr{I}^-$ and the future null infinity $\mathscr{I}^+$ as the *images of the field* φ at $\mathscr{I}^-$ and $\mathscr{I}^+$ and denote them by an appropriate capital letter:

$$\tilde{\varphi}\Big|_{\mathscr{I}^\pm} = \Phi_{\substack{\text{in}\\ \text{out}}} . \tag{5.1.10}$$

The field $\tilde{\varphi}$ in the Penrose space satisfies the equation

$$\left(\tilde{\Box} - \frac{1}{6}\tilde{R}\right)\tilde{\varphi} = 0, \tag{5.1.11}$$

where $\tilde{\Box} = \tilde{g}^{\alpha\beta}\tilde{\nabla}_\alpha \tilde{\nabla}_\beta$, and $\tilde{R}$ is the scalar curvature of the metric $\tilde{g}_{\alpha\beta}$. Having fixed the image Φ_{in} of the field φ in the asymptotically simple space, one can find φ by solving equation (5.1.11) with initial data on the regular null surface $\mathscr{I}^-$, whereby Φ_{out} is determined. In other words, a one-to-one correspondence between the field φ and its images at $\mathscr{I}^-$ and $\mathscr{I}^+$ exists in the asymptotically simple space, provided the asymptotically regular solution exists globally:

$$\Phi_{\text{in}} \leftrightarrow \varphi \leftrightarrow \Phi_{\text{out}}, \tag{5.1.12}$$

The condition of asymptotic regularity plays here the role of radiation condition, and the problem of the classical scattering theory can be formulated as the problem of finding the image of solution φ at $\mathscr{I}^+$, having a specified image at $\mathscr{I}^+$ Note that the asymptotically regular field in the asymptotic region (close to $\mathscr{I}$) is of the form

$$\varphi \sim \Phi\Omega. \tag{5.1.13}$$

In Minkowski spaces in (5.1.4) coordinates, the resulting asymptotic behavior in the wave zone is

$$\varphi \sim \frac{\Phi_{\text{out}}(u, \theta, \varphi)}{r}. \tag{5.1.14}$$

The method presented above is readily generalized to other massless fields [see e.g., Penrose (1965b, 1968), Pirani (1964), Frolov (1979, 1986)].

As the asymptotically flat space has a group of asymptotic symmetries, it is possible to determine such characteristics of massless fields as energy and momentum of the incoming or outgoing flux. Let $\xi^\nu_{(a)}$ (a=0,1,2,3) be the generators of the translation subgroup of the BMS group; $\xi^\nu_{(a)}$ act on $\mathscr{I}^\pm$. The expression for energy (a = 0) and momentum (a = 1,2,3) of the incoming (outgoing) radiation is

$$\overset{a}{P}_{\mathscr{I}^\pm} = -\int_{\mathscr{I}^\pm} \tilde{T}_{\mu\nu}\xi^\nu_{(a)}\tilde{g}^{\mu\alpha}\, d\sigma_\alpha, \tag{5.1.15}$$

where $\tilde{T}_{\mu\nu} = \Omega^{-2}T_{\mu\nu}$ is the metric energy-momentum tensor of the field of interest. It can be readily shown that $\overset{a}{P}_{\mathscr{I}^\pm}$ for asymptotically regular fields in a flat spacetime coincides with the total energy-momentum of the system, provided we define it in

terms of vector Killing fields corresponding to translations. In the general case, expression (5.1.15) for asymptotically flat space can be written in terms of massless field images at $\mathcal{J}^{\pm}$. Thus, we have for the scalar field satisfying equation (5.1.9)

$$P^a_{\mathcal{J}} = \int_{\mathcal{J}} N_a \left[(\partial_u \Phi)^2 - \frac{1}{6} \partial_u^2 \Phi^2 \right] du\, d\omega \tag{5.1.16}$$

where

$$N_a = (1, \sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta), \quad d\omega = \sin\theta\, d\theta\, d\varphi. \tag{5.1.17}$$

Note that for wavepacket-type fields φ with finite energy, the value of $|\partial_u \Phi^2|$ decreases as $|u| \to \infty$ so that after integration by parts, the second term can be dropped

$$P^a_{\mathcal{J}} = \int_{\mathcal{J}} N_a (\partial_u \Phi)^2\, du\, d\omega. \tag{5.1.18}$$

Expressions for energy-momentum of the incoming and outgoing fluxes of other massless fields in terms of field images at $\mathcal{J}^{\pm}$ can be written in a similar manner [see, e.g., Frolov (1986)].

5.2. Event Horizon. Penrose Theorem

Now we can give a rigorous definition to the concept of a black hole. A black hole in asymptotically flat spacetime is defined as such a region that no causal signal (i.e., a signal propagating at a velocity not greater than that of light) from it can reach $\mathcal{J}^+$. The causal curve describing the propagation of such a signal is either smooth (a vector u^μ tangent to it has the property $u_\mu u^\mu \leq 0$) or consists of segments of such curves. Let us define the causal past $J^-(Q)$ of a set Q as the set of points with the following property: for each point there is a future-directed casual curve connecting it with one of the points of Q. The set of points visible to a distant observer coincides with $J^-(\mathcal{J}^+)$ The boundary of this set, $\dot{J}(\mathcal{J}^+)$, denoted by H^+, is called the *event horizon* . The event horizon constitutes the boundary of the black hole. Obviously, not only one but several black holes may exist in a bounded region of spacetime, new black holes may be created, and those already existing may interact and merge. In such cases, $\dot{J}(\mathcal{J}^+)$ is the union of the boundaries of all black holes. If an asymptotically flat space contains no horizon, then all events taking place in this space can be recorded, after an appropriate time, by a distant observer. If an event horizon appeared, it means that a black hole has been born and the strongly enhanced gravitational field has produced qualitative changes in causal structure of spacetime. The enhanced gravitational field prevents signals from escaping, so that the observer can never find out about the events inside the black hole unless he chooses to cross the event horizon and fall into it.

Figures 50a and 50b represents a spherically symmetric black hole produced by

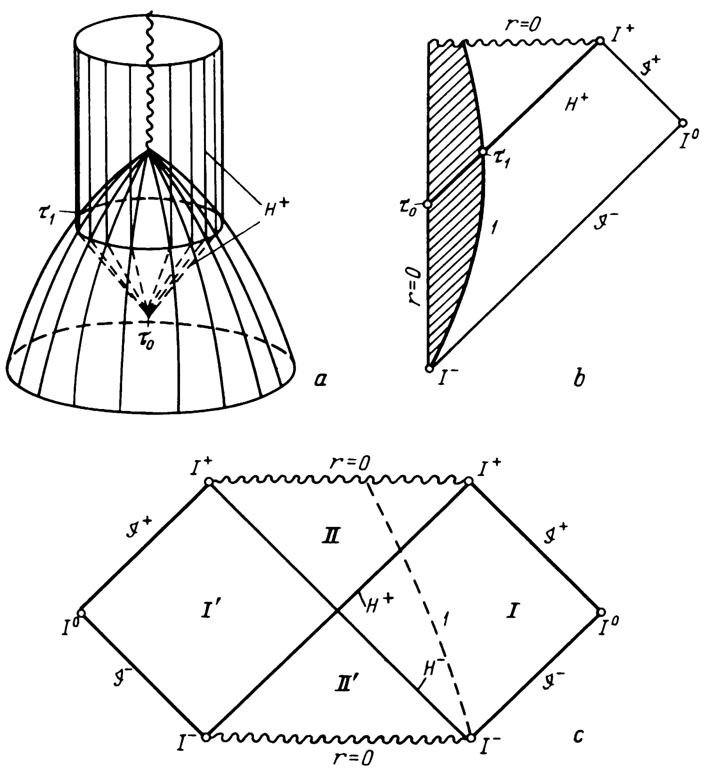


Fig. 50. (a) Spacetime of a spherically symmetric black hole in Finkelstein coordinates. (b) Penrose diagram for the spacetime of a spherically symetric black hole. Line 1 plots the motion of the surface of the collapsing body. The conformal factor is chosen to have line $r = 0$ (the center of the body) vertical. (c) Penrose diagram for 'eternal' black and white holes. Line 1 is the world line of the test particle.

the collapse of a spherical star. We already know that this is the simplest type of black hole. Figure 50a plots the spacetime of this in the Eddington–Finkelstein coordinates. Figure 50b is the Penrose diagram for the corresponding spacetime which can be obtained from the Penrose diagram of the total spacetime of the eternal black hole, shown in Figure 50c, by 'cutting' it along line 1 corresponding to the motion of the surface of the collapsing body, and then 'gluing' on the left the part of the Penrose diagram that describes the metric inside the collapsing body. The last figure shows that the gravitational-radius surface outside the collapsing body coincides with H^+; the region inside H^+ is the black hole. The structure of the infinity of this spacetime (Figure 50b) is identical to that of the infinity of Minkowski spacetime. Note also that the region from which light rays cannot escape to infinity (i.e., the black hole) arises not at the moment τ_1 when the star contracts to a size equal to its gravitational radius, but at an earlier moment τ_0. The event horizon H^+ is formed by signals that, having left the center of the star at a moment τ_0, propagate at the speed of light and reach the surface of the star precisely when it collapses to its gravitational radius.

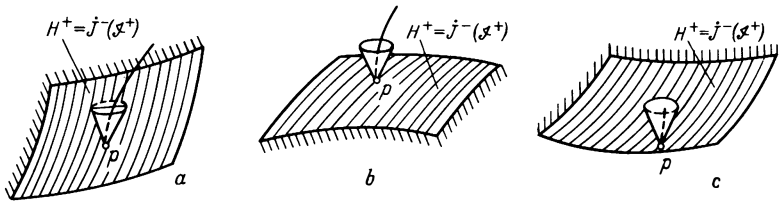


Fig. 51. The regular part of the event horizon is a null surface. The assumption that the event horizon contains a segment where it is (a) time-like, or (b, c) space-like produces a contradiction. The region inside the event horizon is hatched.

The intersection of event horizon with an arbitrary spacelike surface $\Sigma(\tau)$, whose equation is of the type $\tau(x) = \text{const}$, consists in general of a set of closed two-dimensional surfaces $\partial\mathfrak{B}_i(\tau)$ $(i = 1, ..., N)$ that can be regarded as boundaries of the black holes that exist at time τ. The part of $\Sigma(\tau)$ which is bounded by $\partial\mathfrak{B}_i(\tau)$ is called the black hole $\mathfrak{B}_i(\tau)$ at the given moment of time. The number of black holes can be time-dependent.

As we see from Figure 50a, the surface of event horizon is not smooth. Nonregular points can arise on the event horizon after, for example, the fall of matter into the hole. The surface of the event horizon beyond these nonregular points is light-like. This can be illustrated by analyzing the situation presented in Figure 51.

First assume that the surface of the event horizon in a neighborhood of some regular point p of the horizon is time-like (Figure 51a). A contradiction is obtained if we consider the behavior of light rays in a small neighborhood of the point p. Because the structure of spacetime is locally the same as that of Minkowski spacetime, the light rays emitted from p form a future directed local null cone. The time-like part of horizon divides this cone into two parts: the rays go partly into the black hole (to the left of H^+).and partly into the space outside it (to the right of H^+). However, the points of the external space are inside $J^-(\mathcal{J}^+)$; hence; a causal signal can be sent from p and reach $\mathcal{J}^+$. Obviously this is also possible for points to the left of H^+ that are sufficiently close to p. But this conclusion contradicts the assumption that these points lie inside the black hole.

Assume now that the event horizon has a regular spacelike part. If the set $J^-(\mathcal{J}^+)$ lies futureward of this part of H^+ (Figure 51b), then the future-directed light rays emitted at a point p or its small neighborhood inside H^+ can reach $J^-(\mathcal{J}^+)$ and are thus visible to a distant observer. This conclusion is in contradiction with the assumption that the region inside H^+ is a black hole. If the set $J^-(\mathcal{J}^+)$ lies toward the past from the part of H^+ that we consider (Figure 51c), then a point p exists in a small neighborhood of this part of H^+, such that the future-directed null cone with its apex at this point crosses H^+. This means that the event p cannot be seen by a distant observer and p cannot belong to $J^-(\mathcal{J}^+)$, which contradicts the original assumption. Even though these arguments are not absolutely rigorous, they show why the regular segments of the event horizon are light-like surfaces.

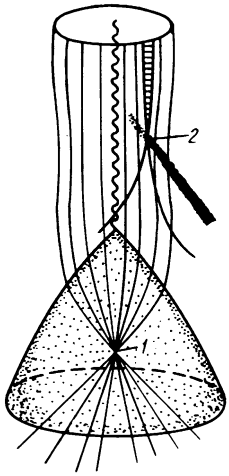


Fig. 52. Illustration to the Penrose theorem on the structure of the event horizon.

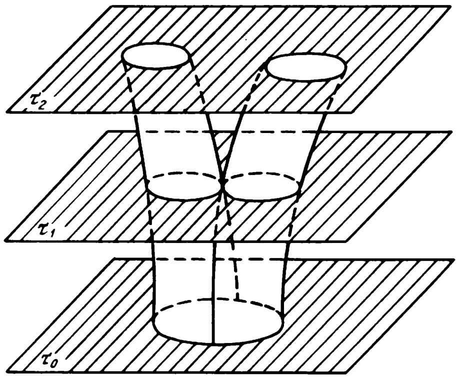


Fig. 53. A black hole cannot break in two or more black holes. The situation shown in the figure (a lone black hole breaking in two) is impossible.

A more detailed description of the structure of the event horizon is provided by the theorem proved by Penrose (1968). According to this theorem, the event horizon is formed by null geodesics (generators) that have no end points in the future. If we monitor the behavior of any generator in the future, we find that it never leaves the horizon H^+ and never intersects another generator. Two versions are possible in the past-directed motion along a chosen generator: either it always lies on H, or this light ray enters H^+ at some point, having left the region $J^-(\mathcal{J}^+)$ outside the black hole. The entry point of this generator at H^+ is the point of its intersection with other generators (the caustics). Only one generator passes through each point of the horizon beyond the caustics. These arguments are illustrated in Figure 52. In this figure, new pencils of rays enter H^+ at caustic 2 when matter falls into the black hole. Caustics 1 corresponds to the point at which the event horizon appears. Caustics also appear when gravitational radiation is incident on the black hole, or when two or more black holes merge.

This situation shown in Figure 53, in which a lone black hole subjected to an external factor breaks into two (or more), is impossible. Indeed, in this process two (or more) points of the black hole's surface which are separated at the initial moment τ_0 move closer and intersect at the moment τ_1 when a bifurcation is formed. This means that at least two generators of the event horizon intersect when extended to the future. This conclusion is in contradiction with the Penrose theorem. The proposition that a black hole cannot split or vanish can be proved rigorously. The proof can be found in the monograph by Hawking and Ellis (1973).

5.3. The Ehlers–Sachs Theorem. Focusing of Light Rays by Gravitational Field

The fact that the event horizon bounding a black hole is a null surface, immediately leads to a number if important properties of black holes. Hence, it is logical to give a more detailed description of the properties of null surfaces in curved spacetime before starting the analysis of black-hole properties.

Let a null surface Γ be given in a spacetime with metric $g_{\mu\nu}$; the equation of the surface in local coordinates x^μ is written in the form $\varphi(x^\mu) = 0$. Then the gradient vector $\varphi_{,\mu}$ is a null vector on Γ: $g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}|_{\varphi=0} = 0$. In general, this relation does not hold outside Γ. But it can be shown [see, e.g., Courant (1962)] that the freedom of choice of φ can be so used that the surface Γ ($\varphi(x) = 0$) belongs to the one-parameter family of null surfaces Γ_c ($\varphi(x) = c$). We assume, with no loss of generality, that this choice was made. Then

$$g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu} = 0. \tag{5.3.1}$$

If we denote $l^\mu = g^{\mu\nu}\varphi_{,\nu}$, then expression (5.3.1) signifies that l^μ *is* a null vector and that it is tangent to the surfaces Γ_c. Furthermore, making use of the property $\varphi_{,\mu;\nu} = \varphi_{,\nu;\mu}$ and of expression (5.3.1), we find $l^\nu l_{\mu;\nu} = l^\nu l_{\nu;\mu} = \frac{1}{2}(l^\nu l_\nu)_{,\mu} = 0$. In other words, the integral curves of the vector field l^μ,

$$\frac{dx^\mu}{dr} = l^\mu, \tag{5.3.2}$$

are geodesics, and r is the affine parameter. If the initial point of integral curve (5.3.2) lies on the null surface Γ, the entire curve belongs to this surface and Γ itself is formed by a two-dimensional family of null geodesics (generators). Let r be the affine parameter along the generators, and $y^a (a = 1, 2)$ be continuous parameters that 'enumerate' these lines. The solution to equation $\varphi = 0$ can then be written in the following parametrized form: $x^\mu = f^\mu(r, y^a)$; also,

$$l^\mu = \frac{\partial f^\mu}{\partial r}, \quad l_\mu \frac{\partial f^\mu}{\partial y^a} = 0, \quad l^\mu l_{\nu;\mu} = 0, \quad l^\mu l_\mu = 0. \tag{5.3.3}$$

From the physical standpoint, the surface Γ describes the propagation of the light wave front, and its generators are light rays perpendicular to the front. If we pick up a narrow beam of light rays the following experiment will provide information on their behavior. Place on opaque object in the path of the beam (perpendicularly to it), and place a screen at some distance from the object, also perpendicularly to the beam. Then the theorem proved by Ehlers and Sachs [Jordan *et al.* (1961), Sachs (1961)] states that all parts of the shadow reach the screen simultaneously. The size, shape, and orientation of the shadow depend only on the position of the screen and are independent of the velocity of the observer. If the screen is at a short distance δr from the object, the expansion and distortion of the shadow are given by the quantities $\theta\delta\rho$ and $|\sigma|\delta r$, where

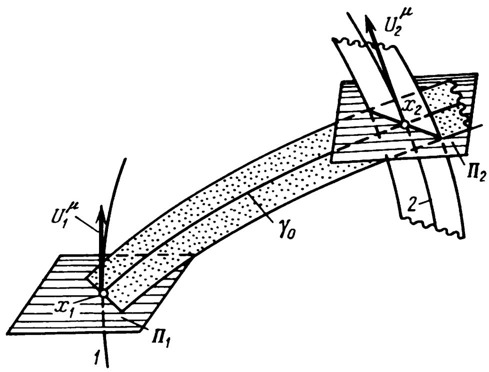


Fig. 54. Illustration of the Ehlers–Sachs theorem on the propagation of light rays.

$$\theta = \frac{1}{2} l^\alpha_{;\alpha}, \quad |\sigma| = [\frac{1}{2} l_{\alpha;\beta} l^{\alpha;\beta} - \theta^2]^{1/2}. \tag{5.3.4}$$

Here we reproduce the main stages of the proof of this theorem, adjusted to the case under consideration. This will enable us to describe in detail a number of important characteristics of null surfaces.[3] Let a light ray γ_0 described by the equation $x^\mu = f^\mu(r, y_0^a)$ intersect the world line of the observer at a point $x_1^\mu = f^\mu(r_1, y_0^a)$; the four-velocity of the observer at this moment is U_1^μ (Figure 54). For the observer, the set Π_1 of the events $x_1^\mu + dx^\mu$ simultaneous to this events satisfies the condition

$$U_{1\mu} dx^\mu = 0. \tag{5.3.5}$$

Choose the affine parameter r so as to satisfy the condition $r = r_1$ at the point of intersection of other rays of the beam and Π_1. If we demand that in addition to (5.3.5) the displacements dx^μ satisfy the condition

$$l_\mu dx^\mu = 0, \tag{5.3.6}$$

then these two conditions jointly define a two-dimensional elements of area perpendicular (in the reference frame of U_1^μ) to the beam of light rays. Let e_a^α ($a = 1,2$) be mutually orthogonal unit vectors and $m^\alpha = 2^{-1/2}(e_1^\alpha + i e_2^\alpha)$. This gives

$$m^\alpha m_\alpha = \overline{m}^\alpha \overline{m}_\alpha = 0, \quad m^\alpha \overline{m}_\alpha = 1. \tag{5.3.7}$$

Assume now that an object is placed in the path of the beam so that a part of the two-dimensional element of area bounded by the curve

$$\zeta^\alpha(\phi) \equiv x^\alpha(\phi) - x_1^\alpha = \zeta(\phi)\, \overline{m}^\alpha + \overline{\zeta}(\phi)\, m^\alpha \tag{5.3.8}$$

is opaque to light rays. A shadow then forms behind the element. Its boundary is defined by the condition

$$x^{\mu} = f^{\mu}(r, y_0^a + \delta y^a(\phi)), \tag{5.3.9}$$

where $y_0^a + \delta y^a(\phi)$ is the value of the parameter y^a of the light ray passing through the point $x^{\alpha}(\theta)$ of the area element, and

$$f^{\mu}(r_1, y_0^a + \delta y^a(\phi)) = x_1^a + \zeta_1^{\alpha}(\phi). \tag{5.3.10}$$

Let the light ray γ_0 extended to a point $x_2^{\mu} = f^{\mu}(r_2, y_0^a)$ intersect there the world line 2 of another observer whose velocity at the moment is U_2^{μ} (see Figure 54). For this observer, the space Π_2 of events simultaneous with x_2^{μ} is defined by the vectors $dx^{\mu} = x^{\mu} - x_2^{\mu}$ which satisfy the relation $U_{2\mu}dx^{\mu} = 0$. We now use the ambiguity $(r \to r' = A(y^a)(r - r_1) + r_1)$ in choosing the affine parameter in order to have $r = r_2$ for all light rays in the beam at the points of intersection with Π_2. It is readily verified that a two-dimensional area element $x^{\mu} = f^{\mu}(r_2, y_0^a + \delta y^a)$, which describes the position of wavefront at the moment x_2^{μ} in the reference frame of U_2^{μ}, is orthogonal to the direction of the light ray whose tangent vector is

$$l_2^{\mu} = \frac{\partial f^{\mu}}{\partial r}(r_2, y_0^a).$$

We have therefore proved that all parts of the shadow simultaneously reach the screen placed perpendicularly to the ray beam.

The size, shape, and orientation of the image on the screen are uniquely determined if the scalar products $\delta_1 x^{\mu}\ \delta_2 x_{\mu}$ are known for an arbitrary pair of vectors connecting the points x_2^{μ} with the points where the light rays with the parameters $y_0^a + \delta y_1^a$ and $y_0^a + \delta y_2^a$ strike the screen, for

$$\delta_i x^{\mu} = \frac{\partial f^{\mu}}{\partial y^a}(r_2, y_0^a)\ \delta y_i^a.$$

It is not difficult to verify that the vectors $\delta_i x^{\mu}$ are transformed in the transition to a different reference frame $U_2'^{\mu}$ by the formula $\delta_i' x^{\mu} = \delta_i x^{\mu} + a l^{\mu}$, while the scalar products remain unchanged:

$$\delta_1 x^{\mu} \cdot \delta_2 x_{\mu} = \delta_1' x^{\mu} \cdot \delta_2' x_{\mu}. \tag{5.3.11}$$

We have thus demonstrated that the size, shape, and orientation of the shadow are independent of the observer's velocity.

For the sake of convenience, we take for U_2^{μ} the vector obtained from U_1^{μ} by its parallel transport along γ_0, since the characteristics of the image on the screen are independent of the choice of observer. The result of the parallel transport of a vector m^{μ} will be denoted by m_2^{μ}. Parallel transport preserves the orthogonality of m^{μ} to the vectors l^{μ} and U^{μ}; hence, m_2^{μ} and $\overline{m}_2^{\mu}$ stretch the two-dimensional element of the area of the screen orthogonal to the light beam (in the reference frame U_2^{μ}). Let us define a vector

$$\zeta_1^\alpha(r) = \frac{\partial f^\alpha(r, y_0^a)}{\partial y^a}\, \delta y^a$$

which connects the point of the ray γ_0 with the affine parameter r to the point having the same affine parameter on the light ray γ whose parameters are $y_0^a + \delta y^a$. Using the expression for $l^\alpha (l^\alpha = \partial f^\alpha / \partial r)$ and the symmetry relation

$$\frac{\partial^2 f^\alpha}{\partial r\, \partial y^a} = \frac{\partial^2 f^\alpha}{\partial y^a\, \partial r},$$

we find

$$l^\alpha \zeta^\beta_{1;\alpha} - \zeta_1^\alpha l^\beta_{;\alpha} \equiv l^\alpha \zeta^\beta_{1,\alpha} - \zeta_1^\alpha l^\beta_{,\alpha} = 0. \tag{5.3.12}$$

The vector $\zeta_1^\alpha(r_2) = (\zeta + \delta\zeta)\,\overline{m}_2^\alpha + (\bar{\zeta} + \delta\,\bar{\zeta})\, m_2^\alpha$ corresponds to the point at which the ray γ strikes the screen. We denote by $\zeta^\alpha(r)$ the vector obtained at the point r as a result of parallel transport of the vector $\zeta^\alpha = \zeta_1^\alpha(r_1) = \zeta\,\overline{m}^\alpha + \bar{\zeta} m^\alpha$ along γ_0:

$$l^\alpha \zeta^\beta_{;\alpha} = 0. \tag{5.3.13}$$

Since the vectors ζ_1^α and ζ^α coincide at the initial point r_1 we find for small distances $\delta r = r_2 - r_1$ from the screen that

$$\zeta_1^\alpha = (\delta_\beta^\alpha + l^\alpha_{;\beta}\, \delta r)\zeta^\alpha. \tag{5.3.14}$$

Multiplying both sides of this equality by m_α and denoting

$$\rho = -l_{\alpha;\beta} m^\alpha \overline{m}^\beta,\quad \sigma = -l_{\alpha;\beta} m^\alpha m^\beta, \tag{5.3.15}$$

we obtain

$$\delta\zeta = -(\rho\zeta + \sigma\bar{\zeta})\,\delta r. \tag{5.3.16}$$

The mapping

$$\zeta \to \zeta' = \zeta + \delta\zeta = \zeta(1 - \rho\delta r) - \bar{\zeta}\delta r \tag{5.3.17}$$

thus establishes the relation between the shape of the object and that of the shadow.

If the projected object is a circle with a boundary $\zeta(\phi) = \exp(i\phi)$, the boundary of the shadow is given by the expression

$$\zeta' = (1 - \rho\phi r)\exp(i\phi) - \sigma\,\delta r\, \exp(-i\phi), \tag{5.3.18}$$

which describes an ellipse with semiaxes $a_\pm = [1 - (\rho + \bar{\rho} \mp |\sigma|)\, \delta r]$, whose area is $\pi a_+ a_- = \pi[1 - (\rho + \bar{\rho})\, \delta r]$; hence, the factor $\theta = -\mathrm{Re}\,\rho$ describes the expansion of the linear scale. The shear modules $|\sigma|$ gives the degree of deformation of the circle; it is found from the relation $a_+/a_- = 1 + 2|\sigma|\delta r$

The quantities ρ and $|\sigma|$ which are independent of the choice of vectors m^α and characterize the expansion and shear of light ray congruences, are called *optical scalars*. It should be emphasized that $\rho\delta r$ and $\sigma\delta r$ are invariants while ρ and σ are linearly transformed when the affine parameter changes. It is readily verified that if the null surface in question, $\varphi(x) = 0$, is included into the family of null surfaces $\varphi(x) = c$ and $l_\alpha = \partial_\alpha\phi$, then

$$\rho = -\frac{1}{2} l^\alpha_{\ ;\alpha}, \quad |\sigma|^2 = \frac{1}{2} l_{\alpha;\beta} l^{\alpha;\beta} - \rho^2. \tag{5.3.19}$$

Let us emphasize that since $l_{\alpha;\beta} = \varphi_{;\alpha;\beta} = l_{\beta;\alpha}$, the optical scalar ρ for the congruence of light rays forming the surface Γ is a real quantity: $\rho = \bar{\rho}$. The relations obtained above complete the proof of the Ehlers–Sachs theorem.

If the cross-sectional area of a narrow light beam is denoted by δA, its change is described by the following equation implied by (5.3.18) for $\rho = \bar{\rho}$:

$$\frac{\mathrm{d}}{\mathrm{d}r}(\delta A)^{1/2} = -\rho(\delta A)^{1/2}. \tag{5.3.20}$$

If $\rho > 0$, the cross-sectional areas decreases as the affine parameter increases.

The behavior of optical scalars ρ and σ along the light rays is described by a system of first-order ordinary differential equations. The derivation of these equations is based on using the identity

$$l_{\mu;\alpha;\beta} = l_{\mu;\beta;\alpha} + R^\nu_{\ \mu\alpha\beta} l_\nu. \tag{5.3.21}$$

Multiplying both sides of this identity by $l^\beta m^\mu \overline{m}^\alpha$, then choosing m^μ, for the sake of convenience, to be parallel-transportable along l^μ, and taking into account the geodesic condition $l^\mu l^\alpha_{;\mu} = 0$ and the relation $R_{\nu\beta} l^\nu l^\beta = 2R_{\nu\mu\alpha\beta} l^\nu l^\beta m^\mu \overline{m}^\alpha$, we arrive at the following equation:

$$\frac{\mathrm{d}\rho}{\mathrm{d}r} = \rho^2 + \sigma\bar{\sigma} + \Phi, \tag{5.3.22}$$

where $\Phi = 1/2 R_{\alpha\beta} l^\alpha l^\beta$. The equation for σ is derived in a similar manner:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}r} = \sigma(\rho + \bar{\rho}) + \Psi \tag{5.3.23}$$

where $\Psi = C_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma m^\delta$, and $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor [see (A.4)]. [4]

If we assume that there is a test beam, formed of light rays, for which the condition $\rho = \sigma = 0$ is satisfied at the initial moment, then part of the curvature Φ (for $\Psi = 0$) acts as a lens free of astigmatism (σ stays equal to zero) while part of the curvature Ψ (for $\Phi = 0$) acts as a purely astigmatic lens (ρ stays equal to zero).

Relation (5.3.22) makes it possible to prove the following important proposition.

Focusing Theorem. Let $\Phi \geq 0$ and let the inequality $\rho \equiv \rho_0 > 0$ be met at some point $r = r_0$ of the light beam. Then the beam reaches the focal point at some finite distance $r - r_0 \leq \rho_0^{-1}$ from this point where the beam cross-sectional area drops to zero.

To prove this proposition, it is sufficient to make use of the following relation:

$$\frac{d^2}{dr^2}(\delta A)^{1/2} = -(\sigma\bar{\sigma} + \Phi)(\delta A)^{1/2}. \tag{5.3.24}$$

This formula is obtained from (5.3.20) by differentiation and by using (5.3.22). Since the right-hand side of this equality is nonpositive, for $r > r_0$ we have $d(\delta A)^{1/2}/dr \le -\rho_0(\delta A)^{1/2}(r = r_0)$; hence, $(\delta A)^{1/2}$ vanishes at the value of the parameter r which satisfies the inequality $0 < r - r_0 \le \rho_0^{-1}$.

If the gravitational field is described by Einstein's equations, then the condition $\Phi \ge 0$ is equivalent to the relation $T_{\mu\nu}l^\mu l^\nu \ge 0$. This condition is satisfied, for example, if the energy-momentum tensor describing the matter and field distributions satisfies the *weak energy condition* (see Appendix), that is, the energy density $T_{\mu\nu}u^\mu u^\nu$ in the reference frame of an arbitrary observer ($u_\mu u_\mu = -1$) is nonnegative. To prove that $\Phi \ge 0$ follows from the weak energy condition, it is sufficient to consider the limiting case of $\alpha(u)u^\mu \to l^\mu$.

There are grounds to assuming that the weak energy condition always holds when matter and physical fields are described within the framework of *classical theory*. This means that any energy-momentum flux across the null surface has a focusing effect on light rays.

5.4. Hawking's Theorem. Cosmic Censorship Conjecture

The Penrose theorem implies that the event horizon is a null surface whose generators never intersect when extended into the future. The caustics at the horizon, corresponding to the formation of new bundles of generators ($\rho = -\infty$) can arise as a result of the falling of matter or radiation into a black hole, or upon the collision and merging of black holes, or when external source fields act on the black hole. A number of important propositions on the general properties of black holes can be derived from these peculiarities of the event horizon considered together with the general theorems on null surfaces, proved earlier.

Let us consider an infinitely narrow pencil of generators of the event horizon. Let the cross-section of this pencil at a point with the affine parameter r be δA (r). Note that in view of Ehlers–Sachs theorem, $\delta A(r)$ is independent of a special choice of a local observer who measures it, and thus can be regarded as invariant. Assume that the cross-sectional area begins to decrease at some point r_0 and the energy-momentum tensor describing the matter and the physical fields surrounding the black hole (and possibly falling into it) satisfies the weak energy condition. The focusing theorem then holds that the generators in the pencil must intersect at a finite value of the affine parameter. In order to match this result to the Penrose theorem, we have to conclude that either there is a physical singularity at the horizon and the generators of the horizon reach it before intersecting, or that the assumption is wrong that the cross-sectional area of the pencil of generators may

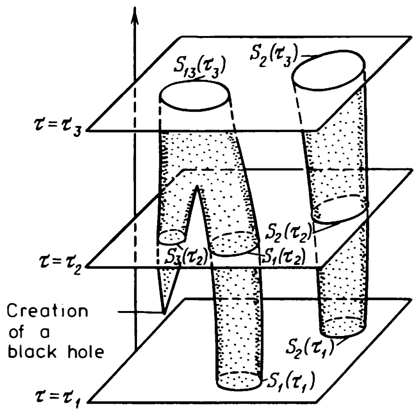


Fig. 55. Possible processes with black holes (illustration to Hawking's theorem). The planes τ_1, τ_2, τ_3 are spatial sections at the corresponding moments of time; $S_a(\tau_i)$ is the area of the black hole a at the instant of time τ_i. Two black holes can merge into one and a new black hole can emerge. The area of a lone black hole does not decrease with time. Hawking's theorem states that the total area of the surface of black holes at a moment τ is a nondecreasing function of time.

start to decrease. In other words, the weak energy condition and the assumption that there are no singularities that can be encountered by the event horizon, imply that the cross-section of the pencil of generators of the event horizon cannot diminish with them. Hawking (1971b, 1972a) proved the theorem that the cross-section of a pencil of generators does not decrease with time even if the condition of no singularities at the event horizon is replaced with the condition of no singularities visible from $\mathcal{I}^+$. Such singularities (visible from $\mathcal{I}^+$) are said to be *naked.* More rigorously, the condition of the absence of naked singularities is formulated as the condition of the existence of a regular space-like surface Σ, such that all causal curves that come out of $\mathcal{I}^+$, necessarily intersect Σ when extended into the past. The existence of such a surface guarantees that if initial data is fixed on it so as to describe completely the state of particles and fields, then the evolution of the system in the region visible from $\mathcal{I}^+$ is uniquely determined. This is equivalent to stating that there are no singularities visible from $\mathcal{I}^+$. In Hawking's terminology, such spaces are *asymptotically predictable.*

Therefore, on the one hand, if we assume that there are no singularities (either at the event horizon or beyond it), then the cross-section of each pencil of generators of the event horizon is nondecreasing in time. On the other hand, if the event horizon contains caustics where new pencils of generators are formed, the horizon cross-section increases. In other words, the sum of areas $S(t)$ of the surfaces of the black holes $\mathcal{B}_i(\tau)$ is a nondecreasing function of 'time' τ. (We assume that $l^\mu \tau_{,\mu} < 0$, that is, the cross-section of the event horizon at a later instant τ corresponds to larger values of the affine parameter along each generator.) A similar conclusion on a nondecreasing area is also valid for an individually taken black hole $\mathcal{B}_i(\tau)$. These results are stated in the theorem proved by Hawking (1971b, 1972a) (Figure 55).

We will briefly discuss the assumptions used to prove the theorem. There are two of them: (1) naked singularities are absent, and (2) the weak energy condition is satisfied. Penrose formulated a hypothesis (1969) known as the 'cosmic censorship conjecture', which states that a naked singularity (i.e., a singularity visible to distant observers) cannot evolve from a regular state of the system under any physically reasonable assumptions concerning the properties of the matter.[5] In other words, the singularities that may appear in evolution of such systems are always hidden from a distant observer by the event horizon.

So far this conjecture has not been proved. Difficulties arise immediately we try to give a more rigorous definition of what a singularity is.[6] Examples can be readily given of a spherical collapse of matter (dust or fluid) in which nonhomological motion of layers forms singularity surfaces outside the event horizon, on which the density of matter tends to infinity [Yodzis *et al.* (1973, 1974), Christodoulou (1984)]. Infinite curvature is also possible when the contraction of a star is accompanied with radiation that carries away some of its mass; complete 'burning' of the star occurs at the moment of collapsing to a point, and the surface of the star is outside the gravitational radius during the entire compression phase [the event horizon is not formed in this case, see Steimüller *et al.* (1975), Lake and Hellaby (1981), Kuroda (1984b)].

Although such examples do contradict the cosmic censorship conjecture, they seem to be rather artificial. Then again, numerical modeling of stellar collapse and an analysis of small deviations from a spherically symmetric collapse indicate that naked singularities are not formed in real physical conditions. A rigorous proof of the cosmic censorship conjecture (and a rigorous formulation of the conditions under which it is valid) remains to be one of the important problems in black hole physics [see the review of Israel (1984), the bibliography therein, and also Caderni and Calvani (1979), Calvani (1980), Glass and Harpaz (1981), Yang (1979), Lake (1979), Israel (1985)].

As for the weak energy condition, the following has to be emphasized: Although this condition seems to be satisfied in the analysis of a black hole interaction with matter and fields in the framework of classical theory, it may be violated (and is indeed violated) when quantum effects are taken into account.[7] Therefore, Hawking's black hole area theorem is directly applicable only to processes for whose description quantum effects are negligible.

5.5. Trapped Surfaces, Apparent Horizon, R- and T-Regions

The boundary of a black hole is the event horizon H^+. At first glance, the definition of a black hole as a region inside the event horizon is very natural. However, if we consider the processes that are possible during the formation of a black hole or in the course of its subsequent evolution, it becomes clear that this definition actually describes not quite what it was meant to do.

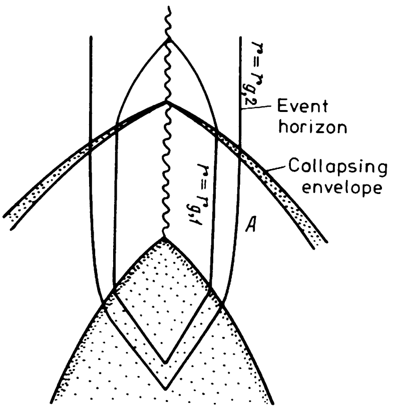


Fig. 56. Nonstatic spherically symmetric black hole.

Indeed, let us imagine that a spherical envelope of mass ΔM falls onto a spherical black hole of mass M some time after it was formed (Figure 56). It would seem that the boundary of the black hole before the envelope reached it was the gravitational radius $r_{g,1} = 2M$, and after the fall the boundary widens and becomes $r_{g,2} = 2(M + \Delta M)$. In fact, the statement that the boundary of the black hole before the fall of ΔM was the $r_{g,1}$ surface is incorrect. Indeed, after the fall of ΔM into the sphere of radius $r_{g,1}$, the null geodesics running along $r_{g,1}$ converge and go into the singularity (see Figure 56). These geodesics are not the boundary of the region from which the rays do not escape to infinity. This region is wider, it is bounded by the rays A shown in Figure 56. Before the fall of the mass ΔM, these rays ran outward from $r_{g,1}$ and somewhat diverged. If ΔM did not fall in, the rays would escape to the infinity $\mathcal{J}^+$. But the falling mass eliminates the divergence and makes the rays go along $r_{g,2}$.

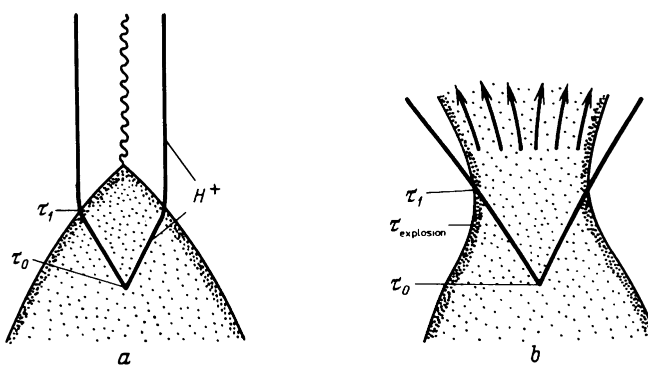


Fig. 57. The position of the event horizon at a given moment of time depends on the entire subsequent evolution of the system. When a spherical collapse forms a black hole (a), the event horizon appears at a moment τ_0. The explosion of the collapsing star occurring after the moment τ_0 may completely preclude the formation of the event horizon (b).

The boundary of the black hole, H^+ is thus determined by not only some specifics of the spacetime at a given moment (say, a strong field in some region) but also by the entire future history (whether a mass ΔM falls or does not fall, etc.). The problem of finding the event horizon H^+ is a problem with final, not initial, conditions.

The following situation is even more impressive. Recall the process of creation of a spherical black hole (Figure 57a). We know that H^+ (and hence, the black hole) is born at the moment τ_0, earlier than the star collapses to the gravitational radius (before τ_1). But imagine that the star explodes at a moment between τ_0 and τ_1, and its matter is thrown out to infinity, that is, no black hole is formed (Figure 57b).[8] Of course, it would be wrong to say that the black hole did exist in the time interval from τ_0 to the explosion because there was no horizon H^+ in this example.

The boundary H^+ thus bounded not so much the region with an especially strong gravitational field (although this field is certainly necessary, or H^+ would not appear at all) as a region with very specific global properties; namely, no rays escape from this region to infinity. It is this property – the invisibility from infinity, the impossibility for particles and light rays to escape – that justifies the name of black hole for this region. In addition, the event horizon is formed by null geodesics, for which a number of strong theorems can be formulated (we gave some of them above). This is another reason for choosing H^+ as the definition for the black-hole boundary.

There is a question, though: Does the spacetime inside a black hole have specific properties which are qualitatively different from those outside the black hole at a given moment, so that it is possible to say that the black hole does exist at this moment, without analyzing the entire infinite future history of the world? We shall now see that, in general, such specific properties do exist.

As we have already mentioned, the following gedanken experiment can detect the

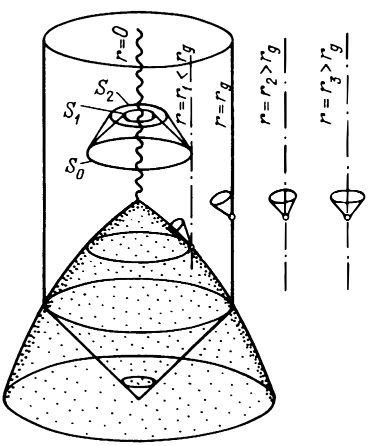


Fig. 58. Position of local null cones inside and outside the black hole. The areas of the inner (S_1) and outer (S_2) radiation wavefronts emitted normally to a two-dimensional surface S_0 inside the black hole are less than the area of S_0.

falling of a spherically collapsing body into a black hole. Let the surface of the collapsing body give a flash at some moment of time. If the body is transparent, then after a short interval of time there are two surfaces, one outside and one inside the body, which correspond to the positions of the outer and inner fronts of the light wave. It is characteristic of the situation inside the black hole that the surface areas of both the inner and the outer wavefronts decrease with time, and the light rays orthogonal to them converge. The gravitational field inside the black hole is so strong that it forces even the outgoing rays that leave the collapsing body to fall centerward (the positions of null cones inside the black hole are shown in Figure 58).

In the general case, closed orientable smooth two-dimensional space-like surfaces, such that both families of null geodesics orthogonal to them converge ($\rho > 0$), are called *trapped surfaces*. The presence of a trapped surface indicates that the gravitational field in the region through which this surface passes is very strong. In asymptotically flat spaces (with asymptotically predictable future),[9] the trapped surface cannot be visible from $\mathcal{J}^+$, unless the weak energy condition is violated [Hawking and Ellis (1973)]. In other words, trapped surfaces lie inside black holes and their existence points to the formation of a black hole.

The presence of a trapped surface is thus the condition sufficient for the existence of a black hole at a given moment of time. We will see that this condition is not the necessary one. Note that the closeness constraint is essential in the definition of the trapped surface. Such a surface surrounding the gravitating center contracts so 'swiftly' that the property described above (convergence) occurs even for light rays going out of this surface. If the closeness constraint is not imposed, all other properties of the trapped surface can be realized without gravitation in Minkowski's flat world. Thus, the intersection of two past null cones for two points p_1 and p_2 in Minkowski spacetime (Figure 59) gives a two-dimensional space-like surface S having the required properties, except for closeness.

Monitoring the behavior of the outgoing radiation is found to be sufficient for solving a number of problems connected with the propagation of signals in the field of a black hole. Consequently, the following definition proves to be convenient:

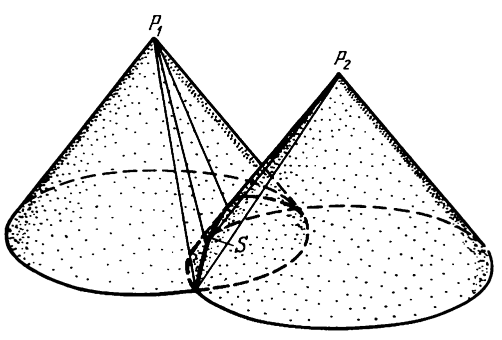


Fig. 59. An example of noncompact two-dimensional surface in Minkowski space. Both orthogonally emerging families of light rays are convergent. Such a surface S is not a trapped one.

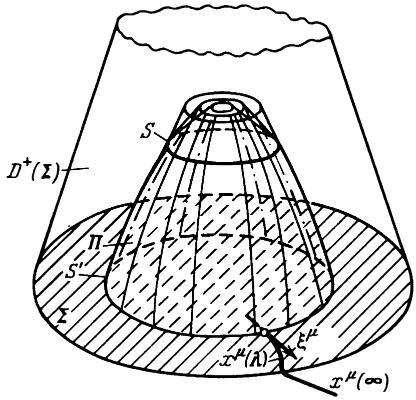


Fig. 60. Illustration of the definition of an outer trapped surface.

The *outer trapped surface* is a compact orientable space-like surface with a property that the divergence of the outgoing null geodesics orthogonal to this surface is nonnegative ($\rho \geq 0$). This definition implies that it is possible to determine in an invariant manner which of the two families of light rays orthogonal to the surface is the outgoing one.

Let the trapped surface S in question arise as a result of the evolution of a system with initial conditions given on the Cauchy surface Σ (Figure 60). Assume for simplicity that Σ is of topology R^3, and consider an arbitrary congruence of smooth time-like lines (the existence of such congruences can be guaranteed, at least in the future Cauchy region $D^+(\Sigma)$ of the surface Σ [Hawking and Ellis (1973)]).[10] The individual curves of a congruence can be treated as world lines of local observers. If we traced past-directed congruences lines passing through the surface S, the points of intersection of these lines with the Cauchy surface Σ form a closed orientable surface S' on Σ. Finding the outward direction on this surface is straightforward. For instance, one can take any smooth curve $x^\mu(\lambda)$ which begins on S' ($x^\mu(0) \in S'$) and reaches the spatial infinity. Then the vector $\xi^\mu = dx^\mu/d\lambda$ for $\lambda = 0$ defines the outward direction at a point $x^\mu(0)$. As the surface Π formed by congruence lines passing through S is orientable, the outward direction ξ^μ defined at one point of this surface determines the outward directions at each of its points, including the points of the original surface S. Such a definition is invariant and can be generalized to the case of asymptotically predictable spaces [Hawking and Ellis (1973)].

We say that a point p lies in the trapped region (briefly, T_--region) if there exists an outer trapped surface passing through this point. In this important particular case of spherically symmetric spaces, a point p belongs to a trapped region if the condition $(\nabla r\ \ \nabla r)_p \leq 0$ is satisfied.

Figure 61a shows a T_--region for the simplest case of the collapse of a spherical dust cloud not followed by a subsequent fall of matter into the black hole. In view of the importance of this concept, the same spacetime is shown again in Figure 61b in Lemaitre coordinates. The metric inside the dust ball can be written in the form

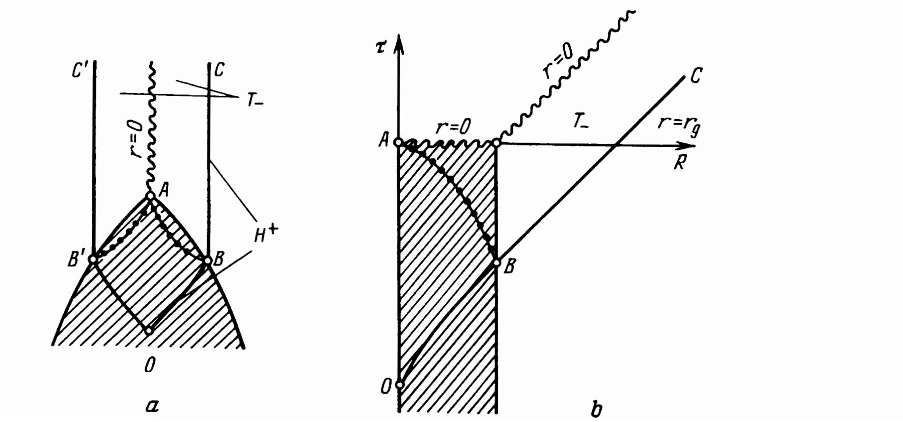


Fig. 61. (a) Position of the T-region in the collapse of a spherical dust cloud; (b) the same spacetime in Lemaitre coordinates.

$$ds^2 = -d\tau^2 + \left[\frac{3}{2}(r_g - \tau)\right]^{4/3} r_g^{-4/3}(dR^2 + R^2\, d\omega^2); \tag{5.5.1}$$

the boundary of the ball corresponds to $R = r_g$. The metric outside the ball, in the vacuum, is of the form

$$ds^2 = -d\tau^2 + \frac{dR^2}{\left(\frac{3}{2}\frac{R-\tau}{r_g}\right)^{2/3}} + \left[\frac{3}{2}(R-\tau)\right]^{4/3} r_g^{2/3}\, d\omega^2. \tag{5.5.2}$$

The boundary of the T_--region in matter is described by the equation

$$\tau + \frac{2}{3} r_g \left(\frac{R}{r_g}\right)^3 = r_g. \tag{5.5.3}$$

This boundary is space-like. The boundary of the T_--region outside the matter, in the vacuum, is r_g:

$$\frac{3}{2}(R - \tau) = r_g. \tag{5.5.4}$$

This boundary is light-like.

The outer component $\partial T_i(\tau)$ of the connected part $T_i(\tau)$ of the intersection of the trapped region and a space-like surface $\tau(x) = \text{const}$ is called the *apparent horizon.*

This is a two-dimensional surface for which the outgoing orthogonal null geodesics have zero divergence ($\rho = 0$). When the weak energy condition is satisfied, the apparent horizon coincides with the event horizon or lies inside it. In stationary

black holes, the apparent horizon and the event horizon coincide (e.g., this is the case for stationary spherically symmetric black holes; see Figure 61).

In the general case, the rays emerging orthogonally to the apparent horizon $\partial T(\tau)$ have zero divergence. In view of Equation (5.3.22), the condition $\rho = 0$ still holds along these rays until they intersect the region where $\Phi > 0$ or $\sigma \neq 0$. In this region ρ becomes positive so that light rays leave the apparent horizon and go into the trapped region. In other words, the other boundary of the T_--region is a null surface in the region where $\sigma = 0$, $\Phi = 0$, but becomes a space-like region where $\sigma \neq 0$ and (or) $\Phi > 0$ (see Figure 61). If the weak energy condition is satisfied, there is always the event horizon beyond the apparent horizon. It must be emphasized, nevertheless, that in general there may be no trapped surfaces inside the event horizon. On the other hand, several connected components of the trapped region can exist inside one black hole. This situation is illustrated in Figure 62 which represents a spherically symmetric black hole which becomes nonstationary for some time because of the matter falling into it. The T_--region lies inside the boundary $EDCBAB'C'D'E'$ shown by the dotted line. The outer boundary of each connected component of the T_--region in the τ = const section forms the apparent horizon. There is no T_--region (no trapped surfaces) in the section $\tau = \tau_1$ = const inside the black hole (inside H^+). This proves that the presence of trapped surfaces in the section τ = const inside the black hole is not the necessary condition for the existence of the event horizon. The section $\tau = \tau_2$ = const contains two connected T_--regions; the inner apparent horizon is $r = r_{g_1}$ and the outer one is $r = r_{g_2}$. A similar situation when several connected components of T_--regions are formed may arise, for example, when two black holes merge.

The following definition will prove to be useful hereafter: A closed orientable smooth two-dimensional space-like surface is said to be *antitrapped* (T_+-region) if both families of null geodesics orthogonal to it are divergent ($\rho < 0$). The formation of T_+-regions is typical for white holes (see Chapter 13). We refer to the region of spacetime that lies beyond both T_- and T_+ regions as the R-region.

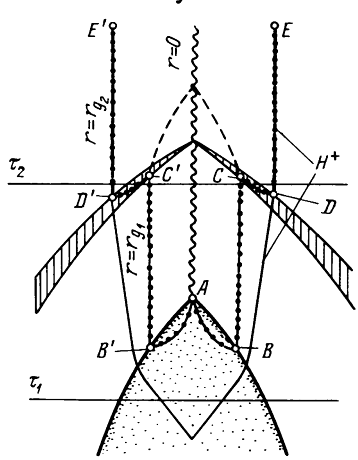


Fig. 62. Accretion of matter onto a spherical black hole.

5.6. Theorems on Singularities Inside Black Holes

We have mentioned in the analysis of spherical collapse that it inevitably produces a singularity, at least in the framework of general relativity. The invariants characterizing the spacetime curvature increase in the course of collapsing, so that the curvature at the center of the collapsing body grows infinitely after a finite time measured by a clock fixed to the body. This situation sets in when the boundary of the T_--region crosses the $r = 0$ line. Further continuation of the world lines of particles and light rays that 'reached' the newly formed singularity becomes impossible. As a result, the incompleteness of a space due to the cutoff of light rays and world lines at the singularity at a finite value of the affine parameter cannot, in principle, be eliminated.

A situation in which the density of matter and tidal forces grow infinitely is also possible when the collapse of a dust ball is described in terms of the conventional Newtonian theory of gravitation. The essential feature is that pressure and small deviations from spherical symmetry, when taken into account, change the picture drastically. The maximal values of matter density and tidal forces (in Newtonian theory, they are analogous to spacetime curvature) become bounded from above. Singularity in Newtonian theory is thus degenerate and unstable in the sense that it arises only in very special situations. Small perturbations are sufficient for eliminating the singularity.

The situation in general relativity is essentially different and the creation of a singularity inside a black hole inevitably proceeds under very general conditions. This conclusion is supported by a number of rigorous theorems.

If one assumes, on the one hand, that the weak energy condition is satisfied and a trapped surface has formed (in other words, that there is a black hole), the areas of the wavefronts of the outgoing and incoming radiation decrease. On the other hand, since the velocity of motion of matter does not exceed the speed of light, the gap between these contracting surfaces always contains all the matter that ever got into this region. The matter will be compressed, its density will increase. It is natural to expect that a singularity or some other 'unpleasantness' will appear.

What 'unpleasantness' could it be? Until now we have interpreted the singularity as infinite curvature of spacetime. This singularity is definitely to be classified as physical because if any world line of a particle runs into this infinity, the line cannot, in principle, be continued beyond it. Here the particle ceases to exist. But this behavior does not exhaust the peculiarities of the spacetime that are connected with a singularity, because of a possible complexity of spacetime topology and the indefiniteness of its metric. For instance, consider the following situation. Let infinite curvature (singularity) exist at some point in spacetime. We cut this singularity, together with a neighborhood, out of spacetime. The remaining manifold has finite curvature only. Can this manifold be considered as free of singularities? Obviously, this conclusion would be incorrect. The point is that the world lines that earlier ran into infinite curvature are now truncated at the boundary of the excised region. This is also a physical irregularity that is also covered by the term 'singularity'. Normally, singularity encompasses not only infinite curvature

but also any end point on the world line of a particle (or a photon) or on a time-like geodesic, provided the line cannot, in principle, be continued beyond this point. It should be stressed that the end point – the singularity – must lie at a finite distance or at a finite value of the affine parameter for a null geodesic. In a more general case, the singularity is thus defined as the incompleteness of world lines in spacetime [for details, see Geroch (1968), Schmidt (1971), Geroch *et al.* (1972), Hawking and Ellis (1973), Clark (1973, 1975, 1976), Geroch *et al.* (1982), Tipler *et al.* (1980)].

Having given these clarifications, we can return to discussing the problem of the inevitability of singularities inside trapped surfaces. The relevant theorem proved by Penrose (1965a) states:

Let the weak energy condition be satisfied and a trapped surface exist in a spacetime admitting a noncompact Cauchy surface Σ. Then this spacetime cannot be complete with respect to null geodesics. In other words, this space contains at least one light ray that cannot be continued and that ends at a finite value of the affine parameter. Hence, the definition given above implies that the singularity exists.

The idea of the proof can be outlined as follows. Consider a set $J^+(S)$ of points that can be connected with S by a causal past-directed curve (Figure 63). Local analysis shows that at points where the boundary $\partial J^+(S)$ of this set is nonsingular, this boundary is light-like and consists of segments of null geodesics that are orthogonal to S at the initial points. If the null generators of $\partial J^+(S)$ have end points, these points coincide with the singularities of $\partial J^+(S)$ (caustics or intersection points). Now it is possible to prove, using the weak energy condition and the convergence of the generators of $\partial J^+(S)$ on the surface S, that each of the light rays emitted orthogonally to S inevitably arrives at a caustic; furthermore, this occurs when the affine parameter does not exceed ρ_S^{-1}, where ρ_S is the maximum value of ρ on S for both families of outgoing rays. (The existence of ρ_S is guaranteed by the smoothness and compactness of the surface S.) Hence, the boundary $\partial J^+(S)$ is compact because it is formed by a compact system of finite closed segments. It can be proved that $\partial J^+(S)$ is a three-dimensional manifold without edge. Note that the proof of compactness of $\partial J^+(S)$ essentially uses the assumption that the spacetime is complete with respect to null geodesics, so that

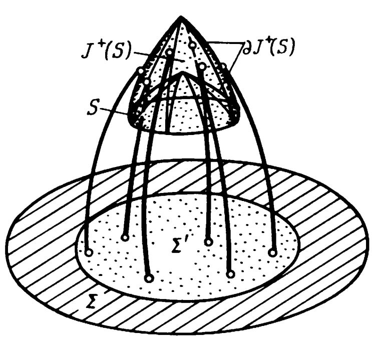


Fig. 63. Illustration to the proof of the Penrose theorem on the singularity inside a black hole.

the generators of $\partial J^+(S)$ are not terminated before arriving at a caustic or an intersection point.

The next step of the proof is to establish that the compactness of $\partial J^+(S)$ contradicts the noncompactness of the Cauchy surface Σ, after which it becomes obvious that the assumption on the completeness of spacetime is incompatible with the rest of the hypotheses of the theorem.

The contradiction is established in the following manner. It can be shown that congruence of time-like curves exists in a spacetime with the Cauchy surface. Since one and only one congruence curve passes through each point of space, and since a time-like curve cannot intersect the null surface $\partial J^+(S)$ more than once, this congruence establishes continuous one-to-one correspondence between $\partial J^+(S)$ and a close subset Σ' of the surface Σ. Note that Σ' cannot coincide with Σ because by assumption Σ is not compact. Hence, Σ' has a boundary in Σ. However, this is in contradiction with $\partial J^+(S)$ being a manifold without edge. The contradiction thus obtained completes the proof of the Penrose theorem on singularity.

It must be emphasized that the condition of noncompactness of the Cauchy surface Σ was used only in proving that Σ' is not identical to Σ. One could demand instead that at least one time-like curve of the congruence does not intersect $\partial J^+(S)$.

We will reproduce here one additional theorem on singularities (in a sense, this is the strongest one among the theorems of this kind), referring the reader interested in exact formulations to the papers of Penrose (1968, 1979), Hawking and Ellis (1973), Misner, Thorne, and Wheeler (1973), Wald (1984), Tipler *et al.* (1980).

Hawking–Penrose Theorem [Hawking and Penrose (1970)]. The spacetime M necessarily contains incomplete time-like or null geodesics that cannot be continued, provided the following conditions are satisfied: (1) the spacetime contains no closed time-like curves; (2) arbitrary time-like unit vectors u^μ obey the inequality $R_{\mu\nu}u^\mu u^\nu \geq 0$; (3) for each time-like or null geodesic with a tangent vector u^μ, there exists a point in which $u_{[\alpha}R_{\beta]\gamma\delta[\varepsilon}u_{\rho]}u^\gamma u^\delta \neq 0$; (4) a trapped surface exists.

All these conditions appear to be fairly reasonable and general. Condition (1) corresponds to our conventional interpretation of causality.[11] Condition (2) signifies that the energy density ε is nonnegative in any physical reference frame and $\varepsilon+3p \geq 0$. Condition (3) is equivalent to stating that we deal with a spacetime of a general type, without special symmetries. Condition (4) is closely connected, as we have already mentioned, with the existence of a black hole. The Penrose–Hawking theorem guarantees that a singularity arises also in the case when a trapped surface is generated, say, in a closed Universe. There would be no noncompact Cauchy surface and the Penrose theorem would be inapplicable.

Note that the above theorems on singularities can be to some extent generalized. They permit extension to the case when the energy dominance conditions are met only on average [for details, see Tipler (1978a, b) and Roman (1986, 1988)].

Chapter 6

Stationary Black Holes

6.1. 'Black Holes Have No Hair'

As a result of the light-like nature of the event horizon enclosing the black hole, the horizon acts as a one-way membrane. Particles and radiation can cross the event horizon from outside and enter the black hole, but escape is forbidden for both particles and radiation. Hence, the processes involving black holes are essentially irreversible. A black hole, if left alone, eventually reaches a stationary state.

The following standard arguments lead to this conclusion. Let collapse produce a black hole, with a nonequilibrium configuration of particles and fields in its exterior. This configuration will immediately begin to rearrange. The rearrangement is accompanied by the radiation of energy to infinity and into the black hole. As the fields and particles outside the black hole originally had finite energies, and since nothing compensates for energy radiated away or absorbed into the black hole, it is to be expected that this process dies down and the black holes becomes stationary, that is, the geometry of spacetime around the black hole will gradually differ less and less from that of stationary space possessing the Killing vector field $\xi^{\mu}_{(t)}$. If the black hole is produced in a collapse slightly deviating from spherical symmetry, this conclusion is supported by the results described in Chapter 3. It is also confirmed by the theorem on the stability of stationary black holes with respect to small perturbations [see Chapter 3 and also Price (1972a, b), Wald (1979a, 1980)] and by the property [proved by Chandrasekhar and Detweiler (1975a), Detweiler (1977, 1979), and Chandrasekhar (1983)] that the gravitational perturbations in the spacetime of a stationary black hole cannot have eigenmodes with purely real frequency (zero damping) (see also Chapters 3 and 4).

It is not difficult to verify that the equilibrium of a given physical field in the neighborhood of the surface of a stationary black hole implies stringent constraints on the admissible configurations of the field [Israel (1971)]. For the sake of simplification, consider a nonrotating black hole described by the Schwarzschild metric

$$ds^2 = -\Phi dt^2 + \frac{dr^2}{\Phi} + r^2 d\omega^2, \quad \Phi = 1 - \frac{2M}{r}. \tag{6.1.1}$$

Let $T^{\mu\nu}$ be the energy-momentum tensor describing the physical field or medium in the neighborhood of this black hole. This system can be in equilibrium if the 'weight' of the field or medium in each element of volume is exactly balanced out by the 'buoyancy' due to the action of the components of the energy-momentum tensor (describing the tension) on the surface enclosing the chosen volume. In the local limit, this peculiar 'Archimedean principle' reduces to the equation (a conservation law)

$$T^{\nu}_{\mu;\nu} = 0, \tag{6.1.2}$$

supplemented with staticity conditions

$$\partial_t T^{\mu\nu} = 0, \tag{6.1.3}$$

and the condition of the zero fluxes

$$T^{rt} = T^{t\theta} = T^{t\phi} = 0. \tag{6.1.4}$$

Under these conditions, equation (6.1.2) holds identically if $\mu \neq r$; if $\mu = r$, it yields the relation

$$\frac{1}{2}\frac{\partial_r \Phi}{\Phi}(T^t_t - T^r_r)$$

$$= \frac{1}{r^2}\partial_r(r^2 T^r_r) + \frac{1}{\sin\theta}\partial_\theta(\sin\theta\, T^\theta_r) + \partial_\theta(T^\phi_r) - \frac{1}{r}(T^\theta_\theta + T^\phi_\phi). \tag{6.1.5}$$

It is readily shown (e.g. by converting the coordinates to $v = t + r^* = t + r + 2M \ln |r - 2M|$, r, θ, φ, which are regular at the horizon H^+) that in the coordinates (t,r,θ,φ) the components of the tensor $T^{\mu\nu}$, which satisfies condition (6.1.3) and is regular on H^+, manifest the following behavior in the neighborhood of $r = 2M$:

$$T^t_t \propto T^r_r \propto T^\theta_r \propto T^\phi_r \propto T^\theta_\theta \propto T^\theta_\phi \propto T^\phi_\phi \propto O(1). \tag{6.1.6}$$

All terms in the right-hand side of (6.1.5) being finite at $r = 2M$, the condition $T^t_t \approx T^r_r$ must be satisfied in the vicinity of the horizon.[1] Otherwise the configuration cannot be static. What happens if this condition is violated? The component T^t_r is then inevitably nonzero, producing a flux of field energy across the horizon: Energy flows until the field rearranges and reaches a possible equilibrium. The characteristic time of this process is of the order $t \sim r_g/c$.

Wheeler summarized the results of a large number of papers devoted to the final states of the black holes and formulated a conjecture that in its evolution to the stationary state, a lone black hole sheds through radiation all those characteristics that radiation can remove. Since for a spin-s massless boson field the radiation is connected with changing the multipole moment l of the system (provided $l \geq s$), Wheeler conjectured that a stationary black hole resulting from a collapse of neutral matter which interacts only gravitationally ($s = 2$), is described by a metric having

only two free parameters, the mass ($l = 0$) and the angular momentum ($l = 1$). This conclusion was made by Doshkevich *et al.* (1965) and Novikov (1969). If the collapsing matter was electrically charged, the appearing stationary metric is described unambiguously by fixing three parameters: M, J, and Q (the electric charge).[2] The absence of the magnetic field of a black hole was proved by Ginzburg (1964) and Ginzburg and Ozernoi (1964).

A lone stationary black hole cannot be source of a massive field because any radiation mode, including $l = 0$, is allowed for such fields so that, according to Wheeler's conjecture, they must all be radiated away in the course of the transition of the black hole to the stationary state. A similar conclusion must hold for scalar massless fields as well.

The above arguments show that Wheeler's conjecture is equivalent to the following proposition: Regardless of the specific details of the collapse or the structure and properties of the collapsing body, the resulting stationary black hole is described by a geometry specified by the parameters M, J, and Q. Wheeler expressed this property of stationary black holes in the following metaphorical form: "Black holes have no hair".

Almost exhaustive proof of Wheeler's conjecture has been obtained by now. In this chapter we summarize the main results relevant to this proof.

6.2. General Properties of Stationary Black Holes

Ample reasons exist for assuming that in the absence of external perturbations and with quantum effects ignored, the final state of any black hole is stationary; hence, it is natural to begin the description of the properties of these final states with an analysis of stationary black holes.

The stationarity of the spacetime signifies that one can introduce in it such coordinates that the coefficients of the metric are independent of the 'time' coordinate. In 'more geometric' terms, this means that the spacetime allows a one-parameter group of motions whose generators are $\xi^\mu \partial_\mu$, where ξ^μ is the Killing vector field satisfying the equation

$$\xi_{(\mu;\nu)} = 0. \tag{6.2.1}$$

As we want the spacetime not to change under translation 'along the time coordinate', it is logical to demand that the vector ξ be time-like, and that $\xi \cdot \xi < 0$. However, it is impossible to guarantee in the general case that $\xi \cdot \xi$ has the same sign throughout the entire spacetime. An asymptotically flat space will be said to be stationary if it admits a Killing vector field ξ^μ which is time-like ($\xi \cdot \xi < 0$) in the neighborhood of $\mathcal{J}^+$ and $\mathcal{J}^-$.

The following two assumptions must be made in order to prove the substantive propositions concerning general properties of stationary black holes.

(1) The spacetime is regularly predictable.

(2) The spacetime is either empty or contains fields described by hyperbolic equations and satisfying the energy dominance condition, namely, that for arbitrary time-like vectors ξ_1^μ and ξ_2^μ the energy-momentum tensor $T^{\mu\nu}$ of the field satisfies the inequality $T^{\mu\nu}\xi_{1\mu}\xi_{2\nu} \geq 0$.

Assumption (1), concerning the general causal structure of the spacetime and thoroughly discussed in the preceding chapter (see note [9] to p. 103), is largely technical. The dominance energy condition (which implies, e.g., the weak energy condition) signifies that an arbitrary observer finds the local energy to be non-negative and the local energy flux to be nonspace-like. Assumption (2) definitely holds for the electromagnetic field [for additional details, see Hawking and Ellis (1973)]. Throughout this chapter we assume that the constraints set above are satisfied.

In a stationary spacetime, the black hole area is independent of time. As a result, the convergence of light rays generating the event horizon is identically zero and, hence, the apparent horizon coincides with the event horizon. Using relations (5.3.20), (5.3.22), and (5.3.23), it is readily ascertained that the weak energy condition ($\Phi \geq 0$) and the constancy of the black hole area imply that the quantities σ, Φ, and Ψ vanish at the horizon surface

$$\sigma|_{H^+} = 0, \quad \Phi|_{H^+} = 0, \quad \Psi|_{H^+} = 0. \tag{6.2.2}$$

The last two equalities can be interpreted as implying the absence of the fluxes of matter and physical fields ($\Phi = 0$) and of gravitational radiation ($\Psi = 0$) across the event horizon.

At each given instant of time, τ, each connected component of the horizon $\partial\mathcal{B}(\tau)$ is compact and simply connected in the stationary spacetime (as it is in the general case). Furthermore, Hawking (1972a) has shown that in the stationary case the topology of the surface of any black hole is that of the two-dimensional sphere S^2. Black hole surface topologies distinct from S^2 are possible if the dominant energy condition is violated [Geroch and Hartle (1982)]. In principle, it cannot be excluded that a stationary spacetime contains several connected components $\partial\mathcal{B}(\tau)$ and, correspondingly, several black holes 'at rest'. Such equilibrium is only possible if the gravitational attraction is compensated for by electromagnetic repulsion (or by repulsion due to forces of a different nature). Thus, if there are several black holes with masses m_i and charges Q_i satisfying the relation $m_i = Q_i/\sqrt{G}$, a system of black holes is in equilibrium [Hartle and Hawking (1972a), Ohta and Kimura (1982)].

In what follows we treat the case of a lone stationary black hole, and restrict the analysis to the region of spacetime exterior to this black hole. In the general case, the entire spacetime of the stationary black hole may contain, together with the event horizon H^+ also the past event horizon $H^- = \dot{J}^+(\mathscr{I}^-)$ (this is readily verified for a Schwarzschild black hole whose Penrose diagram for the entire spacetime is shown in Figure 50c). The region $J^+(\mathscr{I}^-) \cap J^-(\mathscr{I}^+)$ of spacetime outside H^- and H^+ is said to be *exterior*. A property inherent to events occurring in the exterior region is that causal curves can be traced which connect these events with both $\mathscr{I}^-$ and $\mathscr{I}^+$. It

can be proved that in a stationary space the Killing vector field ξ is nonzero everywhere in the exterior region and on the part $H^+ \cap J^+(\mathcal{J}^-)$ of the event horizon [see, e.g., Hawking and Ellis (1973)].

For a more detailed description of the properties of stationary spaces, it is convenient to introduce the following differential invariant ω^α related to the Killing vector field ξ^μ by the formula

$$\omega^\alpha = \xi_{\mu;\nu}\xi_\lambda e^{\mu\nu\lambda\alpha}, \tag{6.2.3}$$

where $e^{\mu\nu\lambda\alpha}$ is a completely antisymmetric tensor. A stationary space is said to be *static* if $\omega^\alpha = 0$. By Frobenius's theorem, the necessary and sufficient condition for the vector field ξ^μ to be orthogonal to a surface is that ω^α vanishes. In other words, if $\omega^\alpha = 0$, two scalar functions, α and t, can be found, such that

$$\xi_\mu = \alpha t_{,\mu}. \tag{6.2.4}$$

In the region of $\xi^\mu \neq 0$, t can be used as one of the coordinates (the time coordinate); we supplement it with three other coordinates x^i. The coordinates x^i are conveniently chosen as follows. Let us fix an arbitrary surface t = const, define the coordinates x^i on it, extend these coordinates to the entire space, demanding that they be constant along the integral curves of ξ^μ. In such coordinates, the metric of a static space is

$$ds^2 = -V^2\, dt^2 + h_{ij}\, dx^i\, dx^j. \tag{6.2.5}$$

It is readily shown, using the Killing equation (6.2.1), that $\partial_t h_{ij} = 0$, and that one can make use of the arbitrariness $t \to t' = f(t)$ in order to satisfy the equalities

$$\alpha = V^2 = -\xi_\mu \cdot \xi^\mu, \quad \partial_t V = 0, \quad \xi^\mu \partial_\mu = \partial_t. \tag{6.2.6}$$

The still remaining ambiguity in the choice of coordinates corresponds to the transformations

$$t \to t' = t + t_0, \quad x^i \to x'^i = f^i(x^j). \tag{6.2.7}$$

Note that since V and h_{ij} are independent of t, metric (6.2.5) is also invariant under time reversal $t \to -t$. The converse is also true, namely, any stationary metric possessing an additional time reversal symmetry $t \to -t$ is static.

An important property of static black holes is that the Killing vector field ξ^μ is time-like throughout their exterior region, while on the part $H^+ \cap J^+(\mathcal{J}^-)$ of the event horizon, which bounds the exterior region, ξ^μ is nonzero, light-like, and directed along the generators of H^+. This last property is easily proved. Namely, from the equality $\xi_{[\mu;\nu}\xi_{\alpha]} = 0$ which follows from the condition $\omega^\alpha = 0$, and from relation (6.2.1) we obtain

$$2\xi_{\alpha;[\mu}\xi_{\nu]} = -\xi_{\mu;\nu}\xi_\alpha. \tag{6.2.8}$$

This equality immediately yields

$$(V^2)_{;[\mu}\xi_{\nu]} = -V^2\xi_{\mu;\nu} \tag{6.2.9}$$

and, hence, the surface $V^2 = 0$ is light-like because the direction of the normal to it, $(V^2)_{;\mu}$, coincides on this surface with that of the null vector ξ^μ. Since $(V^2)_{;\mu}$ and ξ_μ are parallel, we have

$$-\frac{1}{2}(V^2)^{;\mu} = \xi^{\mu;\nu}\xi_\nu = \kappa\xi^\mu \tag{6.2.10}$$

and, hence, ξ^μ is a vector tangent to the null geodesic (to a generator of the surface $V^2 = 0$). These null geodesics, on the one hand, do not emerge on $\mathcal{J}^+$ because they always stay on the surface where $\xi_\mu\xi^\mu = 0$, while on $\mathcal{J}^+$, $\xi_\mu\xi^\mu = -1$. On the other hand, the divergence of the null generators on the surface $V = 0$ is zero. Hence, this surface is the outer trapped surface, and, at the same time, it is the event horizon because space-time is stationary [Vishveshwara (1968)].

If a stationary black hole is not static, the Killing vector field ξ^μ inevitably becomes space-like in a part of the exterior region [Hawking and Ellis (1973)]. This region in which $\xi^2 > 0$ is called the *ergosphere.*

The formation of the ergosphere outside a stationary nonstatic black hole has a number of important physical corollaries. These will be discussed in more detail in the chapters to follow. Here we consider one of them. Recall that by Noether's theorem, the symmetries of spacetime imply the laws of conservation of certain physical quantities. Thus, the uniformity with respect to time implies the conservation of energy. For a particle moving in a stationary spacetime with a Killing vector field ξ^μ, this conserved quantity (energy ε) is written in the form [3]

$$\varepsilon = -p^\mu\xi_\mu, \tag{6.2.11}$$

where p^μ is the four-momentum of a particle. For particles beyond the ergosphere, $\varepsilon \geq 0$ because p^μ is a future directed time-like or null vector. However, some particles of light rays in the ergosphere may satisfy the reversed inequality $\varepsilon < 0$. Obviously, such particles can escape from the ergosphere only by sinking into the black hole. This is impossible unless the ergosphere has common points with the event horizon.

Because of the existence of states with negative energy ε in the ergosphere, the following mechanism of energy extraction from stationary black holes becomes possible [Penrose (1969)]. Imagine that a particle with momentum p_0^μ enters the ergosphere and there decays into a pair of particles with momenta p_1^μ and $p_2^\mu (p_0^\mu = p_1^\mu + p_2^\mu)$, so that $\varepsilon_2 = -p_2^\mu\xi_\mu < 0$, and the particle with momentum p_1^μ escapes from the ergosphere (see Figure 64). The energy of escaping particle, $\varepsilon_1 = -p_1^\mu\xi_\mu = \varepsilon_0 - \varepsilon_2$, is then greater than the energy of the incident particle, $\varepsilon_0 = -p_0^\mu\xi_\mu$; hence, energy can indeed be extracted in this process.

An essential difference between the properties of black holes with $\omega^\mu = 0$ and those with $\omega^\mu \neq 0$ is that, in a certain sense, nonstatic stationary black holes are rotating. The appearance of negative energies of particles moving in the field of a

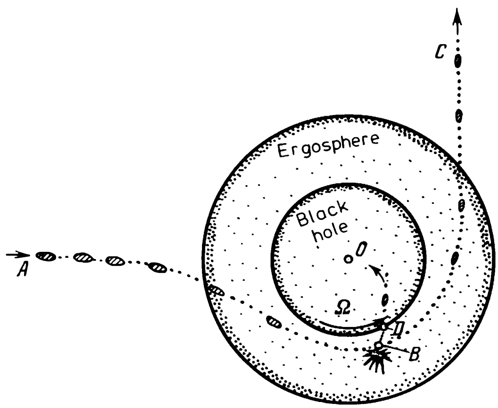


Fig. 64. Penrose process. A body falling from a certain distance (position A) enters the ergosphere of a rotating black hole and explodes at a point B close to the black hole surface in two fragments. One fragment is absorbed by the black hole at a point D (the 'explosion' parameters are chosen so as to have negative energy of this fragment). The other fragment is ejected from the ergosphere (point C) having an energy greater than the energy of the falling body.

rotating black hole can be explained if we take into account the additional gravitational interaction of the angular momentum of this particle and the angular momentum of the rotating black hole. The energy imparted to the particle ejected in the Penrose process is drawn from the rotational energy of the black hole.

Having made these remarks, we can return to discussing the general properties of stationary black holes and consider the mutual arrangement of the ergosphere and the event horizon. In principle, it is possible that the ergosphere does not intersect the horizon, lying completely outside it. However, this situation appears to be unstable [Hawking and Ellis (1973)]. We assume, therefore, that the ergosphere does intersect the horizon in the stationary spacetime describing the final state of a black hole. [For recent discussion of this point, see Carter (1987).] This means that the event horizon comprises points at which the Killing vector field ξ^μ is space-like. Let us show that in this case the black hole is necessarily axially symmetric.

Let S_0 be the surface of a black hole at some moment of time. As we have mentioned earlier, the topology of a stationary black hole surface is that of a two-dimensional sphere, S^2. We denote by S_v those sections of the event horizon that are produced by translating the points of S_0 along the integral curves $x^\mu(v)$ of the field ξ^μ $(\mathrm{d}x^\mu/\mathrm{d}v = \xi^\mu)$ by an amount corresponding to the parameter v (Figure 65). Let a point $p_0 \in S_0$ transform thereby into a point $p_v^\xi \in S$. Denote by p_v^l the point of intersection of S_v and the generator of the event horizon passing through p_0. Since l^μ and ξ^μ are not parallel, the mapping $p_v^l \to p_v^\xi$ is a nontrivial transformation of S_v into itself. As a result of vanishing of convergence ρ and shear σ at the event horizon of a stationary black hole, the spacing between any two points p_v^l and q_v^l on

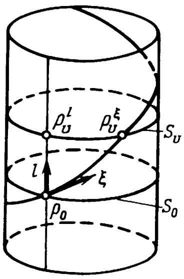


Fig. 65. Stationary rotating black hole is axially symmetric. Illustration of the proof of Hawking's theorem.

S_v is equal to the spacing between the corresponding points p_0 and q_0 on S_0. On the other hand, ξ^μ is the Killing vector field, so that this spacing is also equal to the distance between p_v^ξ and q_v^ξ. Hence, the transformation $p_v^l \rightarrow p_v^\xi$ is a symmetry transformation that maps S_v into itself. The topology of S_v being that of the sphere S^2, all points of S_v, except two (the 'poles'), are moved by the described group of isometry along closed circular orbits.

In other words, the surface of a stationary nonstatic black hole is axially symmetric.

If the metric describing a stationary black hole is analytic,[4] then the axial symmetry of the event horizon implies the axial symmetry of the entire space-time. This result is formulated in the following theorem proved by Hawking (1972a).

Let the ergosphere in a stationary nonstatic space intersect the event horizon at $H^+ \cap J^+(\mathscr{I}^-)$. Then there exists a one-parameter cyclic group of isometries whose generators commute with ξ^μ and whose orbits are space-like in the vicinity of $\mathscr{I}^+$ and $\mathscr{I}^-$.

This theorem is also valid when the metric is nonanalytic in isolated regions outside the horizon.

To summerize the results outlined in this section: The final state of a lone black hole is thus described by a stationary metric. The black hole either does not rotate and its metric is static, or it rotates and the spacetime possesses additional axial symmetry. The two subsequent sections are devoted to proving the so-called *uniqueness theorems*, showing that both static and nonstatic black holes have relatively simple structures. Namely, we analyze the stationary solutions of the Einstein–Maxwell equations and prove that all such solutions that describe a stationary black hole reduce to the Kerr–Newman metric (6.4.33). Furthermore, in the case of no rotation, the solution reduces to the Reissner–Nordström solution. The role played by other physical fields and their 'hair' will be discussed in the last section of this chapter.

6.3. Uniqueness Theorem for Static Black Holes

Let us look at static solutions of Einstein's vacuum equations. We choose the coordinates in the static spacetime as outlined above and write the static metric in the form (6.2.5):

$$ds^2 = -V^2\, dt + h_{ij}\, dx^i\, dx^j,$$
$$i = 1, 2, 3;\ V = V(x^1, x^2, x^3);\ h_{ij} = h_{ij}(x^1, x^2, x^3). \tag{6.3.1}$$

Denote by ${}^{(3)}R_{ij}$ the Ricci tensor of three-dimensional space Σ described by the equation t = const and having the metric h_{ij}. Einstein's vacuum equations are then equivalent to the following equations:

$$h^{ij} V_{:ij} = 0, \tag{6.3.2a}$$

$$V_{:ij} - {}^{(3)}R_{ij} V = 0. \tag{6.3.2b}$$

Here $(\ )_{:i}$ denotes the covariant derivative in the metric h_{ij}.

Assume that our spacetime with metric (6.3.1): (1) is asymptotically flat; (2) has the event horizon; and (3) has no singularities on or outside the event horizon.

Considered in detail, these assumptions imply that

(1) the space Σ is asymptotically Euclidean, that is, coordinates x^i can be chosen, such that

$$h_{ij} = \delta_{ij} + O(r^{-1}),\ \partial_k h_{ij} = O(r^{-2}),\ V = 1 - M/r + \eta, \tag{6.3.3}$$

$$M = \text{const},\ \eta = O(r^{-2}),\ \partial_i \eta = O(r^{-3}),\ \partial_i \partial_j \eta = O(r^{-4})$$

as $r \equiv (\delta_{ij} x^i x^j)^{1/2} \to \infty$.

(2) The function V has zeros on Σ, the set $V(x^i) = 0$ being a connected regular smooth surface.

Rigorously speaking, the points at which $V = 0$, are not covered by the coordinates t, x^1, x^2, x^3 because metric (6.3.1) written in these coordinates has a singularity. The assumption that there exists a regular event horizon implies that a transition to new coordinates makes it possible to extend the metric to that part of the spacetime which contains the event horizon. The surface $V = 0$ can be treated as the boundary of Σ that arises as a result of the passage to the limit $V = \delta =$ const as $\delta \to +0$. The function V satisfies the elliptic equation $V_{:i}{}^{:i} = 0$, and hence, is harmonic. Since $V = 1$ as $r \to \infty$, at finite values of r outside the horizon the values of V are positive and smaller then 1 [on the corresponding property of harmonic functions see, e.g., Jano and Bochner (1953)].

(3) The invariant $\mathfrak{R}^2 = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}$ constructed of the four-dimensional tensor of curvature is finite everywhere on Σ (for $0 \le V < 1$).

The uniqueness theorem for static black holes in vacuum states:

Any static solution of Einstein's vacuum equations satisfying conditions (1)–(3) is spherically symmetric and coincides with the Schwarzschild metric.

Israel (1967) proved this theorem under the following additional condition: (4) equipotential surfaces $V = \text{const} > 0$ are regular simply connected two-dimensional closed surfaces. Later it was proved [see Müller zum Hagen *et al.* (1973)), Robinson (1977)] that this condition which implies, among other things, that $V_{,\alpha} \neq 0$ everywhere in the interval $0 \leq V < 1$, follows from conditions (1)–(3).

The main steps of proving Israel's theorem are as follows. The function V ($V_{,\alpha} \neq 0$) is chosen as a coordinate. The remaining two coordinates θ^2 and θ^3 on the surfaces $V = \text{const}$ are chosen in such a manner that the coordinates lines V be orthogonal to the surfaces $V = \text{const}$. In these coordinates, metric (6.3.1) is written in the form

$$ds^2 = -V^2\,dt^2 + \rho^2\,dV^2 + b_{XY}\,d\theta^X\,d\theta^Y, \tag{6.3.4}$$

where $X, Y = 2, 3$, ρ and b_{XY} are functions of V, θ^2, and θ^3, and Equation (6.3.2a) takes the form

$$\partial_V\left(\frac{\sqrt{b}}{\rho}\right) = 0, \; b = \det(b_{XY}). \tag{6.3.5}$$

Now we define the two-dimensional tensor of exterior curvature K_{XY} of the surface $V = \text{const}$ by the relation

$$K_{XY} = \frac{1}{2\rho}\frac{\partial b_{XY}}{\partial V}. \tag{6.3.6}$$

For the trace of this tensor, $K = b^{XY}K_{XY}$, we can derive the expression $K = \rho^{-1}\,\partial(\ln\sqrt{b})/\partial V$; in view of (6.3.5), this gives

$$\frac{\partial\rho}{\partial V} = \rho^2 K. \tag{6.3.7}$$

It can be shown that Equation (6.3.2b) is equivalent to the following set of equalities

$$\left(\frac{\partial}{\partial V} + \frac{1}{V}\right)K_X{}^Y = -\rho_{|X}{}^{|Y} + \frac{1}{2}\rho\,{}^{(2)}R\,\delta^Y_X - \rho K K^Y{}_X, \tag{6.3.8}$$

$${}^{(2)}R = -K_{XY}K^{XY} + K^2 + \frac{2K}{\rho V}, \tag{6.3.9}$$

$$\partial_X\rho = \rho^2 V(\partial_X K - K_X{}^Y{}_{|Y}), \tag{6.3.10}$$

where ${}^{(2)}R$ is the scalar curvature of the two-dimensional surface $V = \text{const}$, and $(\;)_{|X}$ stands for the covariant derivative in the metric b_{XY}. Equations (6.3.5),

(6.3.6), and (6.3.8) make it possible to find how the unknown functions ρ, b_{XY}, and K_{XY} depend on V, and equations (6.3.9) and (6.3.10) act as constraints: If they are satisfied for one value of V, other equations imply that they are satisfied for all V.

The next step is to find the conditions imposed on the unknown functions by assumption (3). First we rewrite the invariant $\mathfrak{R}^2 = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$, in terms of ρ, b_{XY}, and K_{XY}:

$$\frac{1}{8}\mathfrak{R}^2 = (V\rho)^{-2}\left(K_{XY}K^{XY} + \frac{2\rho_{|X}\rho^{|X}}{\rho^2} + K^2\right). \tag{6.3.11}$$

Equation (6.3.5) implies that $\sqrt{b} = c(x^Y)\rho$; hence, the regularity of the surface $V = 0$ implies that $\rho(V = 0, \theta^2, \theta^3) \neq 0$. The regularity of $\mathfrak{R}^2$ for $V = 0$ now gives

$$K_{XY}(V=0, \theta^2, \theta^3) = 0, \quad \rho(V=0, \theta^2, \theta^3) = \rho_0 = \text{const},$$

$$\lim_{V\to 0}(K/V) = \frac{1}{2}\rho_0 \, {}^{(2)}R(V=0, \theta^2, \theta^3). \tag{6.3.12}$$

If the black hole area $\int_{V=0}\sqrt{b}\, d\theta^2\, d\theta^3$ is denoted by A_0, then by integrating (6.3.5) in V from 0 to 1, and taking into account boundary conditions (6.3.3) and (6.3.12), we find

$$A_0 = 4\pi M\rho_0. \tag{6.3.13}$$

This equality implies, for example, that M is always positive.

The following relations can be obtained from Equations (6.3.5) and (6.3.7)–(6.3.9):

$$\frac{\partial}{\partial V}\left(\frac{\sqrt{b}\,K}{\sqrt{\rho}\,V}\right) = -\frac{2\sqrt{b}}{V}\left[{}^{(2)}\Delta(\rho^{1/2}) + \rho^{-3/2}\left(\frac{1}{2}\rho_{|X}\rho^{|X} + \Psi_{XY}\Psi^{XY}\right)\right], \tag{6.3.14}$$

$$\frac{\partial}{\partial V}\left[\frac{\sqrt{b}}{\rho}\left(KV + \frac{4}{\rho}\right)\right] \tag{6.3.15}$$

$$= -\sqrt{bV}\,[{}^{(2)}\Delta(\ln\rho) + \rho^{-2}(\rho_{|X}\rho^{|X} + 2\Psi_{XY}\Psi^{XY}) + {}^{(2)}R],$$

where

$${}^{(2)}\Delta = (\ )_{|X}{}^{|X} \quad \text{and} \quad \Psi_{XY} = (K_{XY} - \frac{1}{2}Kb_{XY})\rho.$$

The last step of proof consists in integrating relations (6.3.14) and (6.3.15) in V from 0 to 1. Using boundary conditions (6.3.3) and (6.3.12), the identity

$$\int_{V=\text{const}} {}^{(2)}\Delta f \sqrt{b}\ \mathrm{d}\theta^2 \mathrm{d}\theta^3 = 0, \tag{6.3.16}$$

valid for arbitrary smooth function f, and the Gauss–Bonnet theorem,

$$\int_{V=\text{const}} {}^{(2)}R \sqrt{b}\ \mathrm{d}\theta^2 \mathrm{d}\theta^3 = 8\pi, \tag{6.3.17}$$

we arrive at the inequalities

$$\rho_0 \geq 4M,\ A_0 \geq \pi\rho_0^2, \tag{6.3.18}$$

which become equalities if and only if

$$\Psi_{XY} = 0,\ \rho_{|X} = 0 \tag{6.3.19}$$

everywhere on Σ.

Comparing (6.3.18) and (6.3.13), we easily find that these two relations are not in contradiction only if (6.3.18) contains only equality signs and, hence, (6.3.19) holds. These relations show that the relevant vacuum solution to Einstein's equations is spherically symmetric, that is, coincides with the Schwarzschild solution in accordance with Birkhoff's theorem (1923).

A similar uniqueness theorem holds if the condition that Einstein's vacuum equations hold is dropped, and these are replaced with a system of Einstein–Maxwell equations. In this situation, a black hole may be charged. The corresponding unique solution is spherically symmetric and coincides with the Reisner–Nordström metric [Israel (1968)].

6.4. Uniqueness Theorem for Stationary Axially Symmetric Black Holes

Now we will discuss the properties of solutions of the Einstein–Maxwell equations that describe stationary axially symmetric black holes. In addition to the Killing vector field $\xi^\mu_{(t)}$ which is normalized at infinity by the condition $\xi^\mu_{(t)}\cdot\xi_{(t)\mu}, = -1$, there is also a space-like Killing vector field $\xi^\mu_{(\phi)}$ which corresponds to the symmetry of space with respect to rotation. This field commutes with $\xi^\mu_{(t)}$ and has closed integral curves. The field $\xi^\mu_{(\phi)}$ is nonzero everywhere in the exterior region and at the horizon, except on the rotation axis, where $\xi^\mu_{(\phi)} = 0$. If we denote $X = \xi^\mu_{(\phi)}\xi_{(\phi)\mu}$, the condition of regularity of spacetime (i.e. that the spacetime is locally Euclidean) at the rotation axis is satisfied if

$$\left.\frac{X^{,\alpha}X_{,\alpha}}{4X}\right|_{X=0} = 1. \tag{6.4.1}$$

The vector fields $\xi^\mu_{(t)}$ and $\xi^\mu_{(\phi)}$ possessing the properties described above, including the normalization (6.4.1), are uniquely defined in the stationary axially symmetric asymptotically flat space.

In such a space, we can introduce coordinates $t, \phi, x^X (X = 1,2)$ in such a way that the relations

$$ds^2 = -V\,dt^2 + 2W\,d\phi\,dt + X\,d\phi^2 + 2g_{0X}\,dx^X\,dt +$$

$$+ 2g_{\phi X}\,dx^X\,d\phi + \gamma_{XY}\,dx^X\,dx^Y, \tag{6.4.2}$$

where

$$V = -\xi^\alpha_{(t)}\xi_{(t)\alpha},\ X = \xi^\alpha_{(\phi)}\xi_{(\phi)\alpha},\ W = \xi^\alpha_{(t)}\xi_{(\phi)\alpha},$$

$$\xi^\alpha_{(t)} = \delta^\alpha_t,\ \xi^\alpha_{(\phi)} = \delta^\alpha_\phi, \tag{6.4.3}$$

hold, and the functions $V, X, W, g_{0X}, g_{\phi X}$, and γ_{XY} are independent of t and ϕ. Metric (6.4.2) is said to satisfy the *circularity condition* if coordinate transformations can be found that preserve the form (6.4.2) but have the coefficients g_{0X} and $g_{\phi X}$ vanish. In this case the two-dimensional surfaces t = const and ϕ = const are orthogonal to two-dimensional surfaces formed by the integral curves of the fields $\xi^\mu_{(t)}$ and $\xi^\mu_{(\phi)}$. The necessary and sufficient condition for a metric to satisfy the circularity condition is that the following relations are satisfied [see, e.g., Kramer *et al.* (1980)]:

$$e^{\alpha\beta\gamma\delta}\xi_{(\phi)\alpha}\xi_{(t)\beta}\xi_{(t)\gamma;\delta} = 0,\ e^{\alpha\beta\gamma\delta}\xi_{(t)\alpha}\xi_{(\phi)\beta}\xi_{(\phi)\gamma;\delta} = 0. \tag{6.4.4}$$

It can be shown [Kundt and Trümper (1966), Carter (1973a)] that these relations hold if and only if the Ricci tensor $R_{\alpha\beta}$ satisfies the conditions

$$\xi^\delta_{(t)}R_{\delta[\alpha}\xi_{(t)\beta}\xi_{(\phi)\gamma]} = 0,\ \xi^\delta_{(\phi)}R_{\delta[\alpha}\xi_{(\phi)\beta}\xi_{(t)\gamma]} = 0. \tag{6.4.5}$$

Obviously, these conditions are met for the vacuum solutions to Einstein's equations. It is readily verified that they also hold outside sources in electrovac spaces [Carter (1969)]. Therefore, the circularity condition is satisfied in the case of interest here (stationary axially symmetric solutions of the Einstein–Maxwell equations) and the element of length admits the following representation:

$$ds^2 = -V\,dt^2 + 2W\,d\phi\,dt + X\,d\phi^2 + d\gamma^2, \tag{6.4.6}$$

where

$$d\gamma^2 = \gamma_{XY}(x^Z)\,dx^X\,dx^Y.$$

Carter (1969) proved that if the causality condition is met (there are no closed time-like lines), the quantities

$$\rho^2 \equiv VX + W^2 \tag{6.4.7}$$

and X are positive everywhere in the exterior region, except on the rotation axis, where $X = W = 0$, and on the event horizon bounding the exterior region, where ρ^2

vanishes. For a static black hole, $W = 0$ and the equation of the event horizon becomes $V = 0$.

If Einstein's or the Einstein–Maxwell equations are satisfied, the function ρ is harmonic:

$$\frac{1}{\sqrt{\gamma}}\partial_X(\sqrt{\gamma}\,\gamma^{XY}\partial_Y\rho) = 0. \tag{6.4.8}$$

Any two-dimensional metric being conformally flat, we can write $d\gamma^2$ in the form

$$d\gamma^2 = \tilde{U}(x^1, x^2)[(dx^1)^2 + (dx^2)^2]. \tag{6.4.9}$$

However, it is more convenient for the description of the properties of a metric in the neighborhood of the horizon to introduce new coordinates λ and μ which in the asymptotically distant region ($V \to \infty$) are related to the conventional spherical coordinates r and θ by the formulas

$$\lambda \approx r - M, \quad \mu \approx \cos\theta. \tag{6.4.10}$$

Here M is the black hole mass measured by an asymptotically distant observer and $d\gamma^2$ written in these coordinates is

$$d\gamma^2 = U(\lambda, \mu)\, d\gamma_0^2, \tag{6.4.11a}$$

$$d\gamma_0^2 = \frac{d\lambda^2}{\lambda^2 - C^2} + \frac{d\mu^2}{1 - \mu^2}. \tag{6.4.11b}$$

Carter (1971) showed that the coordinates t, λ, μ, and ϕ cover the entire exterior region of a stationary black hole (except for the rotation axis where these coordinates obviously have a singularity). The coordinate ϕ is periodic (with a period of 2π), t varies from $-\infty$ to $+\infty$, μ varies from -1 to $+1$ (the boundary values are attained on the 'north' and 'south' polar axes), and $\lambda > C > 0$ (the value $\lambda = C$ corresponds to the event horizon; for asymptotically distant points, $\lambda \to \infty$). In these coordinates,

$$\rho^2 \equiv VX + W^2 = (\lambda^2 - C^2)(1 - \mu^2), \tag{6.4.12}$$

and the electromagnetic field $F_{\mu\nu}$ outside the sources is written in the form

$$F_{\nu\mu} = \partial_\nu A_\mu - \partial_\mu A_\nu, \quad A_\mu dx^\mu = \Phi\, dt + B\, d\phi \tag{6.4.13}$$

the quantities V, X, W, U, Φ, and B are functions of λ and μ.

Now we will outline the main steps in the proof of the uniqueness theorem for axially symmetric stationary black holes. These steps are as follows.

(1) The solution of the Einstein–Maxwell equation can be reduced, using the method developed by Ernst (1968a, b) [see also Kramer *et al.* (1980)], to solving a system of two second-order elliptic equations for two complex functions of the

variables λ and μ (the Ernst potentials). It is then found that the resulting equations coincide with the equations of motion for a specific Lagrangian.

(2) An analysis is carried out of the conditions imposed on the coefficients of the metric (6.4.6), (6.4.11) and on the components of electromagnetic field, (6.4.13). These conditions stem from the constraint of regularity of spacetime in the neighborhood of the event horizon and on the rotation axis, and also from the assumption that the space is asymptotically flat. These conditions are subsequently reformulated in an equivalent manner as boundary conditions for Ernst potentials at the singular points $\lambda = C$, $\lambda = \infty$, $|\mu| = 1$.

(3) A differential condition relating two arbitrary solutions is obtained using the invariance of the Lagrangian introduced for the problem involved. This condition is used to prove that any two solutions satisfying the derived boundary conditions with fixed values of arbitrary constants are identical.

(4) It is shown that the familiar Kerr–Newman solution, describing a charged rotating black hole, satisfies the boundary conditions mentioned above and contains the required number of arbitrary constants. This procedure establishes that this family of solutions exhausts the set of solutions describing stationary axially symmetric black holes.

The following remark serves as the starting point of the program outlined here. Assume that the functions X, W, Φ, and B corresponding to a certain axially symmetric stationary asymptotically flat solution of the Einstein–Maxwell equations are known. Then the function V for this solution is found from (6.4.12), and the function U can be uniquely determined by solving an equation implied by the complete Einstein–Maxwell system [Kramer *et al.* (1980)].

Then a change of variables Φ, W for new variables E, Y is carried out via the following relations:

$$E_{,\mu} = (X\Phi_{,\lambda} - WB_{,\lambda})/(1-\mu^2),$$

$$E_{,\lambda} = -(X\Phi_{,\mu} - WB_{,\mu})/(\lambda^2 - C^2),$$

$$Y_{,\mu} = (XW_{,\lambda} - WX_{,\lambda})/(1-\mu^2) + 2(BE_{,\mu} - EB_{,\mu}), \tag{6.4.14}$$

$$Y_{,\lambda} = -(XW_{,\mu} - WX_{,\mu})/(\lambda^2 - C^2) + 2(BE_{,\lambda} - EB_{,\lambda}).$$

It can be shown that the original Einstein–Maxwell system of equations provides the consistency conditions for system (6.4.14) and yields four nonlinear partial differential equations for four unknown functions (Ernst potentials) X, Y, E, B which can be obtained by varying the following 'action' functional:

$$S \equiv \int \sqrt{\gamma_0}\, d\lambda d\mu \mathfrak{L}, \tag{6.4.15}$$

where the 'Lagrangian' $\mathfrak{L}$ is

$$\mathcal{L} = (2X^2)^{-1}[(\nabla X)^2 + [\nabla Y + 2(E\,\nabla B - B\,\nabla E)]^2] + 2X^{-1}[(\nabla E)^2 + (\nabla B)^2]. \quad (6.4.16)$$

All operations of contraction and index raising are carried out here using the two-dimensional metric $d\gamma_0^2$ (6.4.11b). In the absence of an electromagnetic field, it is sufficient for solving Einstein's vacuum equations to set $E = B = 0$; the 'Lagrangian' $\mathcal{L}$ then takes the form

$$\mathcal{L} = \frac{(\nabla X)^2 + (\nabla Y)^2}{2X^2}. \quad (6.4.17)$$

Carter (1971, 1973a) proved that the boundary conditions which uniquely determine the solution X, Y, E, B follow from the following assumptions: (a) spacetime is asymptotically flat; (b) spacetime is regular everywhere in the exterior region, including the symmetry axis; (c) the event horizon is a regular surface, that is, it contains no physical singularities. In our case, these assumptions take the following form:

(a) E, B, Y, and $\lambda^{-2}X$ are regular functions of λ^{-1} and μ in the asymptotically distant region (as $\lambda \to \infty$). Their asymptotic behavior is

$$E = Q\mu + O(\lambda^{-1}), \quad B = P\mu + O(\lambda^{-1}),$$

$$Y = 2J\mu(3 - \mu^2) + O(\lambda^{-1}), \quad \lambda^{-2}X = (1 - \mu^2)[1 + O(\lambda^{-1})], \quad (6.4.18)$$

where J, Q, and P are constants that play the roles of angular momentum, electric charge, and magnetic monopole charge of the black hole;

(b) E, B, X and Y on the symmetry axis (as $\mu \to \pm 1$) are regular functions of μ and λ; furthermore, the following conditions are satisfied:

$$E_{,\mu} = O(1), \quad E_{,\lambda} = O(1 - \mu^2), \quad Y_{,\lambda} = O(1 - \mu^2),$$

$$Y_{,\mu} + 2(EB_{,\mu} - BE_{,\mu}) = O(1 - \mu^2),$$

$$B_{,\mu} = O(1), \quad B_{,\lambda} = O(1 - \mu^2),$$

$$X = O(1 - \mu^2), \quad X^{-1}X_{,\mu} = 1 + O(1 - \mu^2). \quad (6.4.19)$$

(c) E, B, X and Y on the event horizon (as $\lambda \to C$) are regular functions of μ and λ, and the following conditions hold

$$E_{,\mu} = O(1), \quad E_{,\lambda} = O(1), \quad B_{,\mu} = O(1), \quad B_{,\lambda} = O(1),$$

$$Y_{,\mu} = O(1), \quad Y_{,\lambda} = O(1), \quad X = O(1). \quad (6.4.20)$$

In the case of no electromagnetic field, setting $E = B = 0$ transforms the above conditions into the boundary conditions for problem (6.4.17).

The next (main) step of the proof is to establish a differential identity that relates two arbitrary stationary axially symmetric solutions. In deriving this identity we follow Mazur (1982). The proof essentially employs the invariance of action (6.4.15)–(6.4.16) under the group SU(1,2) of transformations of field variables.[5] In order to establish this invariance, it is convenient to change the variables X, Y, E, B for new variables ξ, η via the relations

$$-X + iY - E^2 - B^2 = \frac{\xi - 1}{\xi + 1}, \quad E + iB = \frac{\eta}{\xi + 1}. \tag{6.4.21}$$

In these variables the Lagrangian density (6.4.16) is rewritten in the form

$$\mathfrak{L} = 2(1 - \xi\bar{\xi} - \eta\bar{\eta})^{-2} \times$$

$$\times[(1 - \eta\bar{\eta})\, \nabla\xi\, \nabla\bar{\xi} + (1 - \xi\bar{\xi})\, \nabla\eta\, \nabla\bar{\eta} + \xi\bar{\eta}\, \nabla\eta\nabla\bar{\xi} + \eta\bar{\xi}\, \nabla\xi\, \nabla\bar{\eta}], \tag{6.4.22}$$

and the condition of positiveness of X is equivalent to the inequality

$$\xi\bar{\xi} + \eta\bar{\eta} < 1. \tag{6.4.23}$$

Denote by Φ the following nondegenerate matrix constructed of ξ and η:

$$\Phi = (1 - \xi\bar{\xi} - \eta\bar{\eta})^{-1} \begin{pmatrix} 1 + \xi\bar{\xi} + \eta\bar{\eta} & 2\bar{\xi} & 2\bar{\eta} \\ 2\xi & 1 + \xi\bar{\xi} - \eta\bar{\eta} & 2\xi\bar{\eta} \\ 2\eta & 2\eta\bar{\xi} & 1 + \eta\bar{\eta} - \xi\bar{\xi} \end{pmatrix}. \tag{6.4.24}$$

Let

$$j_Y = \nabla_Y \Phi \cdot \Phi^{-1}, \tag{6.4.25}$$

where $\nabla_Y \Phi$ is a matrix obtained from Φ by term-by-term differentiation of its components. A simple procedure verifies that Lagrangian density (6.4.22) admits the following equivalent rearrangement

$$\mathfrak{L} = \frac{1}{4}\, \mathrm{Sp}(j_Y j^Y), \tag{6.4.26}$$

where Sp denotes the operation of finding the trace (spur) of the matrix; the operations with the index Y are carried out using the metric γ_0^{XY}.

Let U be a pseudounitary matrix satisfying the condition

$$U^{+}\eta U = \eta, \quad \eta = \mathrm{diag}(-1, 1, 1), \quad \det U = 1. \tag{6.4.27}$$

Then the matrix

$$\Phi = U \Phi U^{-1} \tag{6.4.28}$$

is of the same form as (6.4.24) for the transformed variables $\tilde{\xi}$ and $\tilde{\eta}$. If the transformation matrix U^Y is independent of x^Y, Lagrangian density (6.4.26)

obviously remains invariant under transformations (6.4.28). In other words, action (6.4.15), (6.4.22) is invariant under the group SU(1,2) of nonlinear transformations $(\xi,\eta) \to (\tilde{\xi}, \tilde{\eta})$ generated by linear representation (6.4.28). In accordance with Noether's first theorem, this invariance implies conservation laws. In our case, they are equivalent to the relation

$$\nabla_\mu(\rho j^\mu) = 0 \tag{6.4.29}$$

holding for the solutions Φ of field equations.

Consider now two arbitrary fields Φ_1 and Φ_2 of type (6.4.24) and form of them a matrix $\Phi = \Phi_1\Phi_2^{-1}$. It is then possible to verify that the following differential identity is correct

$$\mathrm{Sp}\{\Phi[\nabla_X(\rho j_2^X) - \nabla_X(\rho j_1^X)]\} + \nabla_X[\rho\nabla^X(\mathrm{Sp}\Phi)]$$

$$= \rho\,\mathrm{Sp}\{\Phi[j_{1X}j_1^X + j_{2X}j_2^X - 2j_{2X}j_1^X]\}, \tag{6.4.30}$$

where

$$j_i^X = \nabla^X\Phi_i\Phi_i^{-1}. \tag{6.4.31}$$

Identity (6.4.30) serves to complete the proof of the uniqueness theorem. Let (X_1, Y_1, E_1, B_1) and (X_2, Y_2, E_2, B_2) (or Φ_1 and Φ_2) be the solutions describing two stationary axially symmetric black holes and satisfying the regularity conditions (6.4.18)–(6.4.20). Then the first term on the left-hand side of (6.4.30) identically vanishes and the second term vanishes if expression (6.4.30) is integrated on the exterior region $\lambda > C, -1 \le \mu < 1$ and boundary conditions (6.4.18)–(6.4.20) are taken into account. On the other hand, it can be shown [Mazur (1982, 1984)] that the expression on the right-hand side of expression (6.4.30) is nonnegative and therefore vanishes on the solutions Φ_1 and Φ_2. It is then proved that vanishing of the right-hand side of (6.4.30) implies, when boundary conditions (6.4.18)–(6.4.20) are taken into account, that

$$\Phi_1 = \Phi_2, \tag{6.4.32}$$

which implies that only one solution of the field equations in the theory (6.4.15), (6.4.22) satisfies the prescribed boundary conditions. It is proved thereby that any stationary axially symmetric black hole is uniquely defined by fixing the values of four arbitrary parameters: C, J, Q, and P.

To complete the proof, note that the following stationary axially symmetric solution of Einstein–Maxwell equations (the Kerr–Newman solution)

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta\, d\phi)^2 +$$

$$+\frac{\sin^2\theta}{\rho^2}[a\,dt-(r^2+a^2)\,d\phi]^2+\rho^2\left(\frac{dr^2}{\Delta}+d\theta^2\right), \qquad (6.4.33)$$

$$A_\mu dx^\mu = -\rho^{-2}\{Qr(dt-a\sin^2\theta\,d\phi)+P\cos\theta[a\,dt-(r^2+a^2)\,d\phi]\}, \qquad (6.4.34)$$

where $\Delta = r^2 - 2Mr + a^2 + Q^2 + P^2$, satisfies boundary conditions (6.4.18)–(6.4.20) and contains four arbitrary parameters: M, a, Q, and P [related to the parameters J and C by the formulas $J = Ma$ and $C = (M^2 - a^2 - Q^2 - P^2)^{1/2}$, respectively]. Hence, this solution is the most general one, describing a lone stationary axially symmetric black hole in the Einstein–Maxwell theory. It is normally assumed that a black hole has no monopole magnetic charge ($P = 0$). Solution (6.4.33)–(6.4.34) then transforms into solution (4.2.1), (4.8.1), (4.8.2).

This proof of the uniqueness theorem is greatly simplified if the black hole has zero electric charge. This limiting case is obtained by setting $\eta = E = B = 0$ and by denoting by Φ, instead of matrix (6.4.24), the 2×2 matrix obtained from (6.4.24) by deleting the last row and the last column. Identity (6.4.30) then transforms into the identity found by Robinson (1975) in proving the uniqueness theorem for noncharged stationary axially symmetric black holes.

Before considering the possibility of non-electromagnetic 'hair' on black holes, let us have a closer look at the global structure of the Kerr–Newman spacetime.

6.5. Analytic Continuation of the Kerr–Newman Metric Inside the Event Horizon

The stationary metric of a rotating noncharged black hole outside the event horizon was discussed in Section 4.4. We have explained there why the Kerr metric extended into the region within the event horizon cannot describe the spacetime inside the black hole. Of course, the same arguments hold in the general case of charged rotating hole described by the Kerr–Newman metric (see Section 4.8).[6]

Nevertheless, in this section we consider a formal continuation of the Kerr–Newman metric to the region within the event horizon. The following reasons justify this analysis.

First, the structure of this continuation proved to be quite unexpected. An analysis showed that the total spacetime in general relativity may be topologically very complicated. Hypotheses were advanced that travel between spaces can be possible if such a total solution is used, provided there exist formations similar to those described by the total Kerr–Newman solution. In fact, the reliability of such hypotheses became very problematic after this solution has been proved to be unstable inside the event horizon.

Second, in order to prove the instability of the Kerr–Newman solution inside the black hole, one necessarily begins with giving the solution itself and then proves its instability.

The properties of the solution within the event horizon are discussed below in this section; instability is proved in Chapter 12.

The total spacetime of the Kerr–Newman metric is analyzed, in principle, as in the case of the Schwarzschild metric. An additional difficulty arises because of the spherical symmetry. We assume that $M^2 > Q^2 + a^2$, otherwise the solution does not describe a black hole. Recall first of all that in coordinates (4.2.1), (4.8.1), the event horizon lies at $r = r_+ = M + (M^2 - a^2 - Q^2)^{1/2}$ (see Sections 4.3, 4.4, and 4.8). Metric (4.2.1) has a singularity here. The singularity is, in fact, a coordinate one. This can be established by transforming to Kerr coordinates (4.4.2). In the case of a charged black hole, the expression for Δ includes Q^2 [see (4.8.1)]. All curvature invariants are finite at $r = r_+$, and the spacetime has no singularities.

When a metric is considered inside a black hole ($r < r_+$), one has to remember that the coordinates t,r,θ,ϕ need not have a simple meaning of the temporal and spherical spatial coordinates, as they had at infinity in the external space. We have already encountered such behavior when analyzing the Schwarzschild metric (see Section 2.4), where, for example, the variable r became the time coordinate for $r < r_g$. The physical meaning of coordinates in the Kerr–Newman metric is even more complex. The coordinate grid is produced by lines 'traced' in the curved four-dimensional manifold; their physical meaning can be found at each point by considering their orientation with respect to the null cone.

If $r < r_+$ metric (4.2.1), (4.8.1) has singularities at

$$r_- \equiv M - (M^2 - a^2 - Q^2)^{1/2} \tag{6.5.1}$$

and at

$$\rho^2 \equiv r^2 + a^2 \cos^2\theta = 0, \quad \text{i.e., } r = 0,\ \theta = \pi/2. \tag{6.5.2}$$

Singularity (6.5.1) is a coordinate one, like $r = r_+$. Singularity (6.5.2) is a true singularity of spacetime. Figure 66 shows the qualitative structure of the spacetime section t = const, ϕ = const at $r = 0$.

The true singularity in the section t = const is a 'ring' $r = 0, \theta = \pi/2$ in the equatorial plane. Here the spacetime curvature is infinite. If we follow the line θ = const $\neq \pi/2$ (follow it in the mathematical sense), no singularities are met on the way; if r=0, the spacetime is regular and we can go into the region of $r < 0$. The spacetime continues up to $r = -\infty$. However, one cannot think that the section

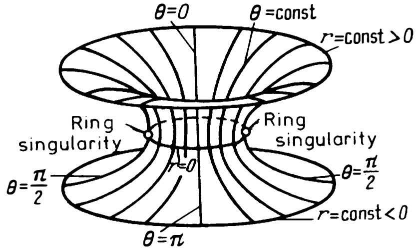


Fig. 66. Qualitative structure of the section t = const, ϕ = const close to $r = 0$.

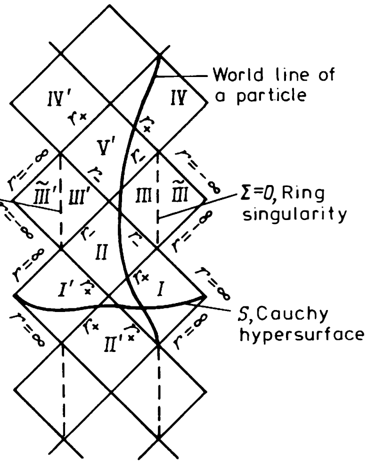


Fig. 67. Penrose diagram for the total Kerr–Newman spacetime.

shown in Figure 66 is space-like. As we see from (4.2.1), the coefficient with $d\phi^2$ becomes negative at r of sufficiently small magnitude and θ close to $\pi/2$; hence, ϕ becomes a time-like coordinate. But ϕ is a cyclic variable with a period of 2π.[7] This means that under these conditions, the section contains closed time lines (lines that lie along the singularity ring and close to it).

A conformal diagram of Figure 67 shows the complete structure of the true analytic continuation of the Kerr–Newman spacetime.[8] A similar diagram for the Schwarzschild spacetime contains four distinct region (see Figure 50c): a white hole, two exterior regions that are asymptotically flat at infinity, and a black hole. The diagram for the Kerr–Newman solution contains an infinite number of regions. Regions I and I' correspond to identical regions of the Schwarzschild black hole. Region II' corresponds to a white hole, and region II, to a black hole. These regions are not bounded by space-like true singularities, as they are in the case of the Schwarzschild solution. Region II connects to regions III and III' across two distinct boundaries $r = r_-$. Each of these regions has a ring singularity discussed above; in each of these regions it is possible to go into the region of $r < 0$ (regions $\widetilde{\text{III}}$ and $\widetilde{\text{III}}$') up to $r \to -\infty$ bypassing the singularity. As $r \to -\infty$, the spaces $\widetilde{\text{III}}$ and $\widetilde{\text{III}}$' become asymptotically flat. Ring singularities $\rho = 0$ manifest themselves in these spaces as 'naked singularities' of negative mass.

Regions III and III' are connected across the boundaries r_- to region V' which is a white hole completely identical in its properties to region II'. Region V', in its turn, is connected across the boundaries r_+ to regions IV and IV' that are completely identical in their properties to I and I', and so on (to infinity).

The timelike line of a particle that fell into the black hole (region II) from the exterior region I continues to the intersection with one of the boundaries $r = r_-$. Only motions to progressively smaller r are possible in the region II. Having crossed $r = r_-$, a particle enters the region III' or III.[9] Here both the motions with decreasing r (down to $r \to -\infty$) and those with increasing r are possible. In the latter

case, a particle crosses the boundary $r = r_-$, enters the region V' (where only the motion with increasing r is possible), and crosses one of the boundaries $r = r_+$, thus emerging in the region IV' and IV. Thus the particle whose world line is shown in Figure 67 can leave 'our' exterior space I and enter another, identical space IV.

Note that the topological structure shown in Figure 67 is conserved in the case of a charged black hole ($Q \neq 0$) even if the hole is not rotating ($a = 0$) (provided $Q^2 < M^2$). The only difference is that in this case the singularity $\rho^2 = 0$ transforms from the ring singularity (in the $t = 0$ section) to a point-like one. Now it becomes impossible to pass by this singularity and enter the region $r < 0$. This is a case when regions III and III' are absent, although it is again possible to pass from I to IV along a time-like world line.

The possibility of such 'journeys' generated a number of exotic hypotheses about the outcome of a real gravitational collapse [Novikov (1966a, b, 1970), De la Cruz and Israel (1967), Bardeen (1968)].

We have already mentioned, however, that owing to the instability of the Kerr–Newman solution inside the black hole, the diagram of Figure 67 has hardly anything to do with reality.

The boundaries $r = r_-$ of region II are known as the Cauchy horizons. This designation reflects the following behavior. If we trace a space-like Cauchy hypersurfaces S in the entire spaces I and I' (and possibly, through parts of regions II' and II) as shown in Figure 67, and fix on this surface the Cauchy data for any fields or particles, then this data defines the evolution of the fields and the motion of the particles only up to boundaries $r = r_-$. Field evolution and particle motion in the regions III and III' can be influenced by sources inside these spaces; the sources are defined indepedently of the data on S.

The following factor constitutes an important property of Cauchy horizon. Figure 67 shows that the later a light signal from the region I reaches II, the closer its world line passes to the boundary r_-. As a result, the world lines of all signals going into the black hole as $t \to \infty$ get 'accumulated' close to r_-. The factor that causes the instability of the Kerr–Newman solution inside a black hole with respect to small perturbations is this concentration of signals along $r = r_-$ (see Chapter 12).

6.6. Generalization of the Uniqueness Theorem to the Case of Nonelectromagnetic Fields

The preceding sections gave uniqueness theorems for the solutions of the Einstein–Maxwell system of equations that describe black holes; these theorems can be considerably generalized and extended to other fields. In this section, we briefly describe the results concerning the 'hair' of black hole when other (nonelectromagnetic) interactions are involved.

Consider a self-consistent system of equations of the gravitational and other physical fields that describe a stationary asymptotically flat space containing a black hole. A general method of analyzing the possibility of existence of solutions in this case was developed by Bekenstein (1972a, b, c). The method is as follows. Let there

be a number of physical fields in addition to the gravitational field. We denote this set by ϕ_A where the domain of the subscript A is determined by the total number of components of the fields under consideration. Let these fields satisfy the equations implied by the action $S[\phi_A] = \int \mathfrak{L}(\phi_A, \phi_{A,\mu})\sqrt{-g}\, d^4x$:

$$\frac{1}{\sqrt{-g}}\left[\sqrt{-g}\frac{\partial \mathfrak{L}}{\partial \phi_{A,\mu}}\right]_{,\mu} - \frac{\partial \mathfrak{L}}{\partial \phi_A} = 0. \tag{6.6.1}$$

Multiply this equation by ϕ_A, take the sum over all A, and integrate over the entire exterior region of the black hole. Denoting

$$b^\mu = \sum_A \phi_A \frac{\partial \mathfrak{L}}{\partial \phi_{A,\mu}} \tag{6.6.2}$$

and making use of Stoke's theorem, we obtain

$$-\int b^\mu d\sigma_\mu + \sum_A \int \left(\phi_{A,\mu}\frac{\partial \mathfrak{L}}{\partial \phi_{A,\mu}} + \phi_A \frac{\partial \mathfrak{L}}{\partial \phi_A}\right)\sqrt{-g}\, d^4x = 0. \tag{6.6.3}$$

Here $d\sigma_\mu$ is an element of hypersurface. Integration is carried out over the entire boundary of the external region of the black hole, that is, the event horizon, the spatial infinity, and the null and the time-like infinities of past and future.

The next relevant fact is that b^μ in the stationary case for massive fields and scalar massless field falls off sufficiently rapidly at the spatial infinity, while the quantity $b^\mu d\sigma_\mu$ vanishes at the horizon and at the time infinity so that the first integral in (6.6.3) drops to zero. If the integrand of the second integral is positive definite, its vanishing implies that the corresponding fields ϕ_A also vanish, which proves the sought result on the absence of 'hair' of this field.

Making use of this method, Bekenstein was able to prove (1972a, b, c) that there cannot be a static black hole with a regular massive scalar, vector or tensor fields outside it that are described by linear equations without sources. A similar result is valid for stationary axially symmetric black holes, provided the metric is assumed to satisfy the circularity condition [Bekenstein (1972c)]. Unfortunately, the proof of the validity of this condition in the general case was not found. This is an obstacle to carrying out the complete proof at the same level of rigorousness as the proof of the uniqueness theorem for electrovac black holes. It can be shown [Bekenstein (1972b, c), Chase (1970)] that a black hole cannot have 'hair' due to a scalar massless field ϕ described by the equation $(\Box - \xi R)\phi = 0$ and vanishing at infinity, provided the value of ϕ at the horizon is finite.[10] A similar result is also proved in the Brans–Dicke scalar-tensor theory [Hawking (1972b)].

The above-described results on the absence of 'hair' are in complete agreement with Wheeler's conjecture because the fields we discuss admit monopole radiation. This situation with the massless Yang–Mills field is different. Wheeler's conjecture does not exclude the possibility of additional monopole degrees of freedom of a

black hole (i.e., charge similar to the electric charge) resulting from the conservation of baryons and leptons. Frolov (1973) found a solution describing a nonrotating black hole having the charge of a Yang–Mills field. Later Yasskin proved (1975) that for every solution of the Einstein–Maxwell equations outside the sources, an ($N-1$)-parameter family of exact solutions of the Einstein–Yang–Mills equations can be constructed for the N-parameter gauge group; this family has the same metric as the original solution has. He also gave the explicit form of all these solutions for a rotating black hole with the charge of the Yang–Mills field. The solutions describing a black hole with this charge (in the presence of a Higgs field), similar to the 't Hooft (1974) and Polyakov (1974, 1975) solution for the monopole, were obtained by Cho and Freund (1975) and Nieuwenhuizen *et al.* (1976).

Hartle (1972) showed that a lepton charge absorbed into a black hole cannot be detected by measuring the forces created by this charge, due to the exchange of a neutrino-antineutrino pair. An interest has been recently revived in proving possible modifications of the 'no-hair' theorem in the framework of supergravity [on black holes in supergravity, see Gibbons (1984)]. Among the results obtained in this field, two are of special interest: the possibility for a black hole to have 'superhair' of the gravitino field, and a new conserved number, the supercharge, appearing as a parameter describing the corresponding solution. This result was obtained by Aichelburg and Güven (1981, 1983a, b) who showed that only the extermal Reissner–Nordström hole admits this generalization in the Kerr–Newman family and obtained the corresponding solution. It should be emphasized, nevertheless, that the physical significance of this solution is not clear because the classical description of the fermion field, corresponding to gravitino 'hair' is rather difficult to justify.

The problem of black hole 'hair' caused interesting developments in multidimensional theories of gravitation[11] of the type of the Kaluza–Klein theory [Dobiasch and Maison (1982), Chodos and Detweiler (1982), Gibbons (1982, 1984), Pollard (1983), Gibbons and Wiltshire (1986), Gibbons (1986), Gibbons and Maeda (1988)]. Such descriptions begin with an assumption that the spacetime is of dimension $n > 4$. The physical spacetime appears as a result of compactification of the 'superfluous' $n - 4$ dimensions. In this n-dimensional space, the original tensor field g manifests itself as a set of physical fields that interact in a prescribed manner with the gravitational field $g_{\mu\nu}$ and with one another. Thus, the simplest version of such theory (for $n = 5$) contains, in addition to the gravitational field, the electromagnetic (A_μ) and a scalar massless (ϕ) fields; note that if the five-dimensional metric is independent of the 'fifth' coordinate, then the five-dimensional action $\int d^5x\sqrt{(-g^{(5)})}\, R^{(5)}$ yields the following four-dimensional action for these fields:

$$\frac{1}{16\pi}\int d^4x\sqrt{-g}\,(R - 2g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi - e^{2\sqrt{3}\,\phi}F_{\alpha\beta}F^{\alpha\beta}). \tag{6.6.4}$$

In this theory neutral rotating black holes are described by a standard Kerr metric. If a black hole has an electric Q or a magnetic P charge, it inevitably has a nonzero scalar charge related unambiguously to Q (or P). The 'scalar hair' of such a black

hole do not contradict the general results given above. The point is that the equation of the scalar field ϕ for action (6.6.4) contains an additional term describing the interaction of ϕ with the electromagnetic field. The factor $F_{\alpha\beta}F^{\alpha\beta}$ in front of $\exp(2\sqrt{3}\,\phi)$ in this equation acts as a sort of source of the scalar field. An explicit form of this metric was given by Gibbons and Wiltshire (1986) for the case of nonrotating black hole. The generalization to rotating black holes is discussed in the same paper and also by Bleyer *et al.* (1987).

To conclude this chapter, we wish to point out the following spectacular feature. If field theories in the flat space and in Einstein's theory are compared, the equations of the latter are found to be much more complicated, so that it is natural to expect that much greater difficulties will be encountered in solving the equations in the latter case. Moreover, the diversity of solutions is also likely to grow if one assumes the existence of black holes and the ensuing changes in the causal structure of space-time. In fact, the reverse is observed, however paradoxical it may seem. The class of possible solutions narrows down dramatically and its complete description becomes possible under certain constraints. The physical reason of this phenomenon is that the gravitational field is universal and acts on any matter possessing energy-momentum. When a black hole is formed, gravitation gets so enhanced that extremely rigid conditions have to be satisfied in its vicinity for physical fields to be in equilibrium; in fact, these conditions are equivalent to eliminating from the field configuration all degrees of freedom capable of propagating. Hence, the general picture is considerably simplified.

Chapter 7

Black-Hole Electrodynamics

Black-hole electrodynamics is defined as the theory of electrodynamic processes that can occur *outside* the event horizon, that is, in the external space accessible for the observation by distant observers.[1]

At the first glance, black-hole electrodynamics is quite trivial. Indeed, we remarked in Section 4.8 that the electromagnetic field of a stationary black hole (of a given mass M) is determined unambiguously by its electric charge Q and rotation parameter a. If the charged black hole does not rotate, its electromagnetic field reduces to the radial electric field of the charge Q and is static. Any multipoles higher that the monopole are absent.

If the black hole rotates, the electromagnetic field has the form (4.8.2). The field is stationary but now the rotation of the black hole, first, induces a magnetic field and, second, distorts the geometry of space and generates the higher-order electric (magnetic) moments of the fields. However, these higher-order moments are determined unambiguously by the quantities M, a, and Q, not being in any way independent, as we find in the case of ordinary bodies.

In astrophysics, the electric charge of a black hole (as of any other celestial body) cannot be high (see Section 4.8). The magnetic field must also be very weak: the dipole magnetic moment of a black hole is $\mu^* = Qa$.

There can be no other stationary electromagnetic field inherent in a black hole. In this sense, the electrodynamics of, say, pulsars is definitely much richer than that of the intrinsic fields of black holes. A pulsar is a rapidly rotating neutron star of approximately solar mass possessing a gigantic 'frozen-in' magnetic field about 10^{12} gauss. Rotation induces high electric fields that 'rip' charges of the surface of the star, accelerate them to high energies, creates a complicated magnetosphere of the pulsar, and generate a host of other related phenomena.

Black holes have neither strong magnetic or electric fields [see Ginzburg (1964), Ginzburg and Ozernoi (1964), and Section 3.4], nor a surface from which charges could be ejected. Complex electrodynamics processes are thus impossible. However, if a black hole is placed in an *external* electromagnetic field and charges in its surroundings can be produced, the situation changes dramatically and complex electrodynamics does appear. It is this aspect that we mean when black hole electrodynamics is discussed.

The case important for astrophysics applications is that of external magnetic (not electric) fields and rarified plasma in which a black hole is embedded.

7.1. Maxwell's Equations

We will consider electromagnetic fields against the background of a given metric, that is, we assume that these fields are not sufficiently strong for affecting the metric. As a rule, this condition is met in astrophysics.[2]

Electrodynamics equations in four-dimensional form, using the tensor $F_{\alpha\beta}$ (see Appendix), can hardly suggest anything to our intuition. They are difficult enough to apply to even modestly difficult concrete problems of physics. Thorne and MacDonald (1982) [see also MacDonald and Thorne (1982), Thorne, Price, and MacDonald (1986)] have rewritten these equations using the '3+1' split for the exterior spacetime of a rotating black hole (see Section 4.2). Their formalism operates with familiar concepts: field strength, charge density, electric current density, and so forth, 'absolute' space and unified 'time'. The equations of electrodynamics are written in a form similar to their form in the flat spacetime in the Lorentz reference frame. As a result, one cannot only use well-developed methods of solving electrodynamics but also 'rely' on conventional notions and the intuition stemming from the experience of solving such problems. Furthermore, the work of Thorne and MacDonald cited above makes use of the so-called 'membrane' interpretation of a black hole. The main point of this approach is that for a distant observer (who does not fall into the black hole), the boundary of a black hole can often be thought of as a thin membrane characterized by special electromagnetic, thermodynamic, and mechanical properties. This interpretation will be described in more detail in Section 7.3 [see also Thorne (1986), Price and Thorne (1986), Thorne, Price, and MacDonald (1986)]. Of course, in reality there is no membrane. This concept needs very careful handling and one must constantly bear in mind that it is only conditional and convenient for solving certain problems. The methods outlined above make it relatively simple to apply black-hole electrodynamics to astrophysics, even when the astrophysicist is not a specialist in relativity. For a review of this field, see Thorne *et al.* (1986).

This section presents the results of the papers of Thorne and MacDonald cited above. All physical quantities introduced below are three-dimensional vectors (or tensors) that are characterized by a position occupied in the 'absolute' three-dimensional space (outside the black hole) and in the absolute unified 'time' t (see Section 4.2). These are quantities that a locally nonrotating observer measures by conventional instruments (see Section 4.3).

We introduce the following notation for electrodynamic physical quantities measured by locally nonrotating observers: $\mathbf{E}$ is the electric field strength, $\mathbf{B}$ is the magnetic field strength, ρ_e is the electric charge density, and $\mathbf{j}$ is the electric current density.

Maxwell's equations are written in the following form:[3]

$$\nabla \mathbf{E} = 4\pi\rho_e, \tag{7.1.1}$$

$$\nabla \mathbf{B} = 0, \tag{7.1.2}$$

$$\nabla \times (\alpha \mathbf{B}) = \frac{4\pi\alpha \mathbf{j}}{c} + \frac{1}{c}[\dot{\mathbf{E}} + \omega \mathfrak{L}_m \mathbf{E} - (\mathbf{E}\nabla\omega)\mathbf{m}], \tag{7.1.3}$$

$$\nabla \times (\alpha \mathbf{E}) = -\frac{1}{c}[\dot{\mathbf{B}} + \omega \mathfrak{L}_m \mathbf{B} - (\mathbf{B}\nabla\omega)\mathbf{m}]. \tag{7.1.4}$$

Here $\alpha = \mathrm{d}\tau/\mathrm{d}t$ [see (4.3.13)]; **m** is a vector of length $A^{1/2} \sin\theta/\rho$ directed along the coordinate lines of ϕ (this is the Killing vector reflecting the axial symmetry of spacetime; far from the black hole, $\mathbf{m} = r \sin\theta\, \mathbf{e}_\phi$); $\mathfrak{L}_m \mathbf{E}$ is the Lie derivative of **E** (or **B**) along the vector **m**, describing how **E** varies with respect to the field of **m** [see (A.7)]:

$$\mathfrak{L}_m \mathbf{E} \equiv (\mathbf{m}\nabla)\mathbf{E} - (\mathbf{E}\nabla)\mathbf{m}, \tag{7.1.5}$$

(this derivative equals zero when the origin and end of the vector **E** are 'glued', in a displacement by $\mathbf{m}\, d\phi$, to the vectors $\mathbf{m}\, d\phi$); ω is the angular velocity of rotation (in time t) of locally nonrotating observers [see (4.3.10)]. A dot denotes differentiation with respect to t and ∇ is the operator of covariant derivative in the curved 'absolute' space.

Equations (7.1.1) – (7.1.2) have familiar form, while that of Equations (7.1.3) – (7.1.4) is slightly unusual. The following differences are evident. A function α has appeared because the physical time flows differently at different points of space while the equations are written in terms of the global 'time' t (recall that the acceleration of free fall, **F**, is related to α in the reference frame of locally nonrotating observers by the formula $\mathbf{F} = -c^2\nabla \ln \alpha$). Furthermore, the expressions in brackets in (7.1.3) and (7.1.4) are 'Lie-type' derivatives (with respect to time) for the set of locally nonrotating observers that move in the absolute space and for which $\mathrm{d}\mathbf{x}/\mathrm{d}t = \omega\mathbf{m}$. These expressions thus corresponds to total derivative with respect to the time of **E** and **B**, respectively, with the motion of locally nonrotating observers taken into account.

Electrodynamic equations become especially lucid and convenient for the analysis of specific problems if written in the integral form [see, e.g., Pikel'ner (1961)]. Here we give only one of such integral expression (we will need it later) for the external space of the black hole, namely, Faraday's law:

$$\oint_{\partial A^*(t)} \alpha\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right) \mathrm{d}\mathbf{l} = -\frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}\oint_{A^*(t)} \mathbf{B}\, \mathrm{d}\boldsymbol{\Sigma}. \tag{7.1.6}$$

Here $\mathrm{d}\boldsymbol{\Sigma}$ is the vector of a surface element, with the vector length equal to the surface area of the element; $A^*(t)$ is a two-dimensional surface that does not intersect

the horizon and is bounded by the curve $\partial A^*(t)$; $d\mathbf{l}$ is an element of $\partial A^*(t)$; and $\mathbf{v}$ is the physical velocity of $A^*(t)$ or $\partial A^*(t)$ relative to locally nonrotating observers.

7.2. Stationary Electrodynamics in the Case of Axial Symmetry. Force-Free Fields

A rotating black hole and the space beyond are stationary and axially symmetric. The motion of matter around a black hole can very often be regarded, with high accuracy, as stationary and axisymmetric as well. It is then natural to assume that the electromagnetic field too has these properties.

In this section we assume that these conditions are satisfied,[4] Then the derivatives with respect to time t and the derivatives $\mathcal{L}_m$ of vectors vanish. Thus, only the last terms survive in brackets in (7.1.3) and (7.1.4).

It is found that under stationarity and axial symmetry, the directly measured values of $\mathbf{E}$, $\mathbf{B}$, ρ_e, and $\mathbf{j}$ are expressible in terms of three arbitrary scalar functions that can be chosen in the following manner.

Let ∂A^* be a closed coordinate line for constant r and θ in the absolute three-space, and A^* be a two-dimensional surface, bounded by ∂A^*, that does not intersect the black hole.

Then these arbitrary functions are:

(1) total current inside the loop ∂A^* (taken with reversed sign):

$$I \equiv -\int_{A^*} \alpha \mathbf{j}\, d\mathbf{\Sigma}, \tag{7.2.1}$$

where $d\mathbf{\Sigma}$ is the surface element (assumed to be positive when oriented in the direction from $\theta = \pi$ to $\theta = 0$);

(2) total magnetic flux across ∂A^*:

$$\Psi \equiv \int_{A^*} \mathbf{B}\, d\mathbf{\Sigma}; \tag{7.2.2}$$

(3) electric potential

$$\mathcal{A}_0 \equiv -\alpha\Phi - \omega \mathbf{A}\mathbf{m}, \tag{7.2.3}$$

where Φ is the scalar and $\mathbf{A}$ is the vector potentials ($\mathbf{m}$ has been defined in Section 7.1).

The quantities I and Ψ depend only on the choice of the position of the loop ∂A^* but are independent of the shape of A^* (we assume the black hole to have zero magnetic charge).

Before expressing $\mathbf{E}$ and $\mathbf{B}$ in terms of I, Ψ, and $\mathcal{A}_0$ we decompose the fields into poloidal (superscript P) and toroidal (superscript T) components that are perpendicular and parallel to the vector $\mathbf{m}$, respectively:

$$\mathbf{E}^T \equiv \left(\frac{A \sin^2 \theta}{\rho^2}\right)^{-1} (\mathbf{Em})\mathbf{m}, \tag{7.2.4}$$

$$\mathbf{B}^T \equiv \left(\frac{A \sin^2 \theta}{\rho^2}\right)^{-1} (\mathbf{Bm})\mathbf{m}, \tag{7.2.5}$$

$$\mathbf{E}^P = \mathbf{E} - \mathbf{E}^T, \tag{7.2.6}$$

$$\mathbf{B}^P = \mathbf{B} - \mathbf{B}^T. \tag{7.2.7}$$

Faraday's law (7.1.6) and the stationarity condition imply

$$\mathbf{E}^T = 0. \tag{7.2.8}$$

Equation (7.1.2) and the condition of axial symmetry of $\mathbf{B}$ (yielding $\mathcal{L}_m\mathbf{B} = 0$) imply

$$\nabla\mathbf{B}^T = 0, \tag{7.2.9}$$

$$\nabla\mathbf{B}^P = 0, \tag{7.2.10}$$

that is, the poloidal and toroidal magnetic lines of force can be treated independently (as not terminating anywhere).

Current density $\mathbf{j}$ is also decomposed into the poloidal ($\mathbf{j}^P$) and toroidal ($\mathbf{j}^T$) components.

Now we can give expressions for all electromagnetic quantities in terms of I, Ψ, and $\mathcal{A}_0$:

$$\mathbf{E}^P = \alpha^{-1}\left(\nabla\mathcal{A}_0 + \frac{\omega}{2\pi c}\nabla\Psi\right), \tag{7.2.11}$$

$$\mathbf{E}^T = 0, \tag{7.2.12}$$

$$\mathbf{B}^P = -\frac{\mathbf{m} \times \nabla\Psi}{2\pi\left(\frac{A \sin^2 \theta}{\rho^2}\right)}, \tag{7.2.13}$$

$$\mathbf{B}^T = -\frac{2I\mathbf{m}}{\alpha c\left(\frac{A \sin^2 \theta}{\rho^2}\right)}, \tag{7.2.14}$$

$$\mathbf{j}^P = \frac{\mathbf{m}\times\nabla I}{2\pi\alpha\left(\dfrac{A\sin^2\theta}{\rho^2}\right)}, \tag{7.2.15}$$

$$j^T \equiv \mathbf{j}^T\mathbf{m}\left(\frac{A\sin^2\theta}{\rho^2}\right)^{-1/2} \tag{7.2.16}$$

$$= \left(\frac{A\sin^2\theta}{\rho^2}\right)^{1/2}\frac{1}{4\pi\alpha}\left\{-c\nabla\left[\frac{\alpha\nabla\Psi}{2\pi\left(\dfrac{A\sin^2\theta}{\rho^2}\right)}\right]+\frac{1}{\alpha}\nabla\omega\left(\nabla\mathcal{Q}_0+\frac{\omega\nabla\Psi}{2\pi c}\right)\right\},$$

$$\rho_e = \frac{1}{4\pi}\nabla\left[\frac{1}{\alpha}\left(\nabla\mathcal{Q}_0+\frac{\omega\nabla\Psi}{2\pi c}\right)\right]. \tag{7.2.17}$$

Note that the last three equations can be treated as differential equations for determining $I, \Psi, \mathcal{Q}_0$ (and, hence, **E** and **B** as well) provided the field sources $\mathbf{j}^P$, $\mathbf{j}^T$, and ρ_e are assumed to be given stationary and axisymmetric but otherwise arbitrary functions. Note also that in the stationary and axisymmetric case the current **j** must be prescribed in such a way that the condition $\nabla(\alpha\mathbf{j}) = 0$ (the charge conservation law) is satisfied, that is, $\alpha\mathbf{j}^P$ must be divergence-free.

Consider now the physical conditions in the plasma surrounding the black hole.

In the case of the maximum importance for astrophysics, the conductivity of plasma is so high that the electric field in the reference frame comoving with plasma vanishes and the magnetic lines of force are 'frozen' into the plasma. In this case, the electric and magnetic fields in an arbitrary reference frame are perpendicular to each other (degenerate fields):

$$\mathbf{E}\mathbf{B} = 0. \tag{7.2.18}$$

The situation in which inertial (and gravitational) forces acting on plasma are small in comparison with electromagnetic forces is even more specific. The configuration of fields and currents is then such that currents in the reference frame comoving with plasma are parallel to magnetic lines of force and no Lorentz force acts on moving charges. Such fields are known as *force-free fields*. In an arbitrary reference frame the condition of existence of a force-free field is

$$\rho_e\mathbf{E} + \frac{1}{c}\mathbf{j}\times\mathbf{B} = 0. \tag{7.2.19}$$

We assume in this section that condition (7.2.19) [and, hence, (7.2.18)] is satisfied in the neighborhood of black holes. [Condition (7.2.19) is violated at the event

horizon; see the next section.] Note also that condition (7.2.19) is definitely violated somewhere in the exterior space of the black hole.

Indeed, the outer magnetic field in normal conditions survives in the space around the black hole because the ends of magnetic lines of force are 'frozen' into the sufficiently dense massive plasma that exists somewhat further away and that has 'transported' the magnetic field to the black hole. Condition (7.2.18) is met in this plasma but condition (7.2.19) is not. The gravitation (and inertia) keeps this plasma in the vicinity of the black hole, together with the magnetic field 'frozen' into it. The lines of force of the field pass from the dense plasma into the region of much more rarified plasma where condition (7.2.19) is satisfied. Some of these lines of force go by the black hole and some go through it. This is the case in, for example, the model of widely discussed disk accretion on a black hole (Figure 68).

If condition (7.2.19) was not violated somewhere and the dense plasma did not counteract the spreading pressure of the magnetic field, this pressure would drive the outward motion of the lines of force together with the rarified plasma.

Condition (7.2.18) is likely to be violated far from the black hole (region 3 in Figure 68), where the magnetic field is sufficiently weak while the inertial forces become relatively large (see the next section).

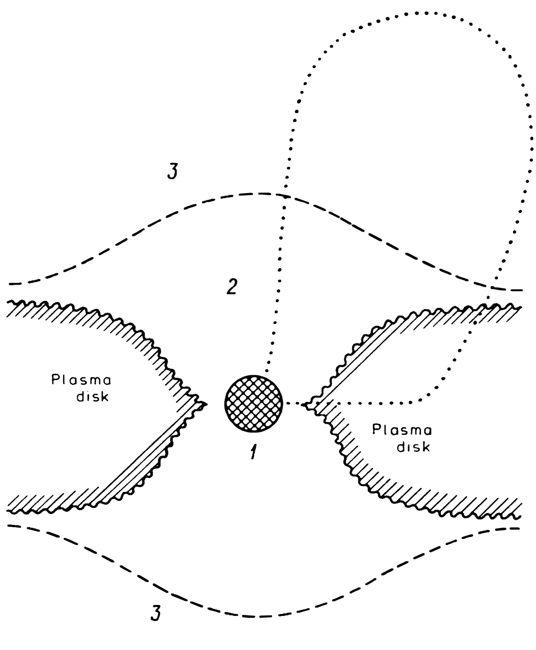


Fig. 68. Schematic representation of disk accretion to a black hole: 1 - rotating black hole, 2 - region of force-free field (7.2.19), 3 - 'acceleration region' where conditions (7.2.18) and (7.2.19) are violated. The dashed curve is the boundary between the regions 2 and 3. The dotted curve is an example of electric current line.

Finally, note that conditions (7.2.18) and (7.2.19) are more approximations. To solve problems concerning the configuration of fields, currents, and charge distributions, it is only necessary that the inequalities

$$|\mathbf{EB}| \ll |\mathbf{E}^2 - \mathbf{B}^2|, \tag{7.2.20}$$

$$\left|\rho_e\mathbf{E} + \frac{1}{c}\mathbf{j} \times \mathbf{B}\right| \ll \left|\frac{\mathbf{j}}{c}\right| |\mathbf{B}|. \tag{7.2.21}$$

are satisfied instead of (7.2.18) and (7.2.19). Small deviations from exact equalities (7.2.18) and (7.2.19) in the neighborhood of a black hole may prove to be essential for a number of astrophysical processes [e.g., see Kardashov *et al.* (1983)].

Let us return to the case of force-free field, assuming conditions (7.2.18) and (7.2.19) to be satisfied exactly in the neighborhood of the black hole. The field **E** is purely poloidal; furthermore, **EB** = 0. Hence, **E** can be presented as the vector product of $\mathbf{B}^P$ by a vector $-\mathbf{v}^F/c$ which is a function of only r and θ and is parallel to **m**:

$$\mathbf{E} \equiv \mathbf{E}^P = -\frac{\mathbf{v}^F}{c} \times \mathbf{B}^P. \tag{7.2.22}$$

Recall that **E** and **B** are fields measured by locally nonrotating observers. Then (7.2.22) implies that the observer moving at a velocity $\mathbf{v}^F$ with respect to locally nonrotating observers measures only the magnetic field. The electric field is zero for this observer owing to the Lorentz transformation. Therefore, $\mathbf{v}^F$ can be interpreted as the linear velocity of the points of a magnetic lines of force with respect to locally nonrotating observers. Field **E** is completely induced by this motion.

If the vector $\mathbf{v}^F$ is written in the form

$$\mathbf{v}^F = \left(\frac{\Omega^F - \omega}{\alpha}\right)\mathbf{m}, \tag{7.2.23}$$

then Ω^F is angular velocity of the points on the lines of force of the poloidal magnetic field in the 'absolute' space. It can be shown that each line of force revolves around the black hole as a whole at the velocity Ω^F which is constant in t in 'absolute' space.

The surface obtained when a magnetic line of force rotates around the symmetry axis is called the *magnetic surface*. The quantity Ψ is obviously constant on this surface , hence, Ω^F is a function of Ψ: $\Omega^F = \Omega^F(\Psi)$. The quantity $\mathcal{Q}_0$ is now also a function of Ψ:

$$\frac{d\mathcal{Q}_0}{d\Psi} = -\frac{\Omega^F}{2\pi c}.$$

It can be shown that condition (7.2.19) specifies the dependence of I on Ψ.

Equations for j^T, (7.2.16), and for ρ_e, (7.2.17), get somewhat simplified

$$j^T = \left(\frac{A\sin^2\theta}{\rho^2}\right)^{1/2} \frac{1}{8\pi^2\alpha}\Big[-c\nabla\left(\frac{\rho^2\alpha\nabla\Psi}{A\sin^2\theta}\right) +$$

$$+\frac{1}{\alpha c}(\Omega^F-\omega)\,\nabla\Psi\,\nabla(\Omega^F-\omega) - \frac{1}{\alpha c}(\Omega^F-\omega)\frac{d\Omega^F}{d\Psi}(\nabla\Psi)^2\Big] \tag{7.2.24}$$

$$\rho_e = -\frac{1}{8\pi^2 c}\nabla\left[\left(\frac{\Omega^F-\omega}{\alpha}\right)\nabla\Psi\right]. \tag{7.2.25}$$

The most important and salient fact is that I, Ψ, and Ω_0 are not any more arbitrary and independent, hence, $\rho_e j^T$, and the divergenceless part of αj^P are equally not arbitrary and independent as they would be if stationary and axial symmetry were the only constraints. All arbitrariness in choosing them has now been eliminated: The necessary (and sufficient) condition for a force-free field to exist is that Ψ satisfy the equation called the *stream equation*:

$$\nabla\left\{\frac{\alpha\rho^2}{A\sin^2\theta}\left[1-\frac{(\Omega^F-\omega)^2}{\alpha^2c^2}\frac{A\sin^2\theta}{\rho^2}\right]\nabla\Psi\right\} +$$

$$+\frac{(\Omega^F-\omega)}{\alpha c^2}\frac{d\Omega^F}{d\Psi}(\nabla\Psi)^2 + \frac{16\pi^2\rho^2}{\alpha c^2 A\sin^2\theta} I\frac{dI}{d\Psi} = 0. \tag{7.2.26}$$

Therefore, if $\Psi, \Omega^F(\Psi)$, and $I(\Psi)$ are chosen so as to satisfy (7.2.26), then **E** for a region with a force-free field is found from (7.2.22) after substituting into it (7.2.23) and (7.2.13); **B** is found from (7.2.13) and (7.2.14), $\mathbf{j}^P$ from (7.2.15), and $\mathbf{j}^T$ and ρ_e from (7.2.24) and (7.2.25), respectively.

7.3. Boundary Conditions at the Event Horizon. Membrane Interpretation and 'Stretched' Horizon

Black-hole thermodynamics treats only processes outside the event horizon. Correspondingly, the solution of electrodynamic equations requires that boundary conditions far from the black hole are to be supplemented with boundary conditions on its surface. Formally, this situation is identical to that arising in pulsar electrodynamics where boundary conditions on the surface of the neutron star must also be specified.

Nevertheless, the two situations are principally different. In contrast to a neutron star a black hole has no material surface that differs from the surrounding space. Formally, the construction of the complete picture of the electromagnetic field dictates that we solve Maxwell's equations for the entire spacetime, that is, both outside and inside the black hole, without any special conditions at the event

horizon. The corresponding conditions have to be referred to the singularity inside the black hole.

It is clear, however, that we are interested only in the processes outside the black hole and remember that the region of spacetime inside it cannot affect these processes, a correct formulation can be given to the problem of electrodynamics outside the black hole, with specially adjusted boundary conditions at the event horizon. The event horizon generators are null geodesics and the same lines are the characteristics of Maxwell's equations, hence, the problem is to be solved with boundary conditions given on a family of characteristics.

The appropriate boundary conditions were formulated by Znajek (1978) and Damour (1978). It was found that they can be presented in a very clear form; namely, we can assume that the surface of the black hole has a *fictitious* surface electric density σ^H charge that compensates for the flux of electric field across the surface, and a *fictitious* surface electric current $\mathfrak{I}^H$ that closes tangent components of the magnetic fields at the horizon. This interpretation is used in the *membrane formalism* [Thorne (1986), Thorne *et al.* (1986)].

MacDonald and Thorne (1982) formulated the conditions at the horizon as follows:

Gauss' law

$$\mathbf{En} \equiv \mathbf{E}_{\perp} \rightarrow 4\pi\sigma^H, \tag{7.3.1}$$

where $\mathbf{n}$ is the unit vector of outward normal to the black-hole surface, and the symbol $\rightarrow$ indicates the approach to the black-hole horizon along the trajectory of a freely falling observer;

charge conservation law

$$\alpha\mathbf{jn} \rightarrow -\frac{\partial\sigma^H}{\partial t} - {}^{(2)}\nabla\mathfrak{I}^H, \tag{7.3.2}$$

where ${}^{(2)}\nabla$ is the two-dimensional divergence at the horizon;

Ampere's law

$$\alpha\mathbf{B}_{\parallel} \rightarrow \mathbf{B}^H \equiv \left(\frac{4\pi}{c}\right)\mathfrak{I}^H \times \mathbf{n}, \tag{7.3.3}$$

where $\mathbf{B}_{\parallel}$ is the magnetic field component tangent to the horizon;

Ohm's law

$$\alpha\mathbf{E}_{\parallel} \rightarrow \mathbf{E}^H \equiv R^H\mathfrak{I}^H, \tag{7.3.4}$$

where $\mathbf{E}_{\parallel}$ is the electric field component tangent to the horizon; $R^H \equiv 4\pi/c$ is the effective surface resistance of the event horizon ($R^H = 377$ ohm).

The factor α in conditions (7.3.2) – (7.3.4) reflects the slowdown in the flow of physical time for locally nonrotating observers in the neighborhood of the black hole.

The values of $\mathbf{E}^H$ and $\mathbf{B}^H$ at the horizon are finite, and $\alpha \to 0$. Hence, taking into account the conditions given above, we arrive at the following properties of fields at the horizon:

$$\mathbf{E}_\perp \text{ and } \mathbf{B}_\perp \text{ are finite a tthe horizon,} \tag{7.3.5}$$

$$|\mathbf{E}_\parallel| \text{ and } |\mathbf{B}_\parallel| \text{ generally diverge at the horizon as } 1/\alpha, \tag{7.3.6}$$

$$|\mathbf{E}_\parallel - \mathbf{n}\times\mathbf{B}_\parallel| \propto \alpha \to 0 \text{ at the horizon.} \tag{7.3.7}$$

Condition (7.3.7) signifies that for locally nonrotating observers, the electromagnetic field at the horizon acquires (in the general case) the characteristics of an electromagnetic wave sinking into the black hole at infinite blue shift.

The conditions listed above make it possible to imagine quite clearly how electromagnetic processes will effect the properties of the black hole, slowly varying its parameters (the change is slow because we have assumed from the beginning that the electromagnetic field is relatively weak; see Section 7.1).

The change in the angular momentum $\mathbf{J}$ of a black hole is equal to the total flux of the electromagnetic field angular momentum across the horizon (all quantities are considered in the global time t). In the differential form, this condition is written as follows:

$$\mathrm{d}J = (\sigma^H \mathbf{E}^H + (\mathfrak{I}^H/c) \times \mathbf{B}_\perp)\mathbf{m}\ \mathrm{d}\Sigma^H\ \mathrm{d}t. \tag{7.3.8}$$

Here $\mathrm{d}\Sigma^H$ is an element of the horizon area. The change in the mass M of the black hole is given by the following expression:

$$\mathrm{d}M \cdot c^2 = \{\Omega^H[\sigma^H \mathbf{E}^H + (\mathfrak{I}^H/c) \times \mathbf{B}_\perp]\mathbf{m} + \mathbf{E}^H \mathfrak{I}^H\}\ \mathrm{d}\Sigma^H\ \mathrm{d}t. \tag{7.3.9}$$

The first term in the braces describes the change in the rotational energy of the black hole and the second term gives the change due to the 'heating' of the black hole by the surface current.

We thus find that the boundary conditions at the event horizon, described in this section, make it possible to model a black hole for electrodynamic problems in the exterior space as an imaginary sphere with specific electromagnetic properties. The sphere can carry surface charge and electric currents. As we already mentioned, this approach is called the membrane formalism.

This visually clear picture helps greatly in solving specific problems.

We wish to emphasize again that there are no real sphere, no charges, and no current at the black-hole boundary. Note also that the fields $\mathbf{E}$ and $\mathbf{B}$ that a locally nonrotating observer measures at the event horizon, differ drastically from $\mathbf{E}^H$ and $\mathbf{B}^H$ that appear in the boundary conditions: the reason is the factor α in the definition of $\mathbf{E}^H$ and $\mathbf{B}^H$ [see (7.3.3) and (7.3.4)]. This factor appears (recall the formulation of Maxwell's equations) as a result of using the 'global' time t.

Note here the following important aspects. In the '3+1' split of the black hole spacetime, the $t = \text{const}$ surfaces behave as shown in Figure 69.

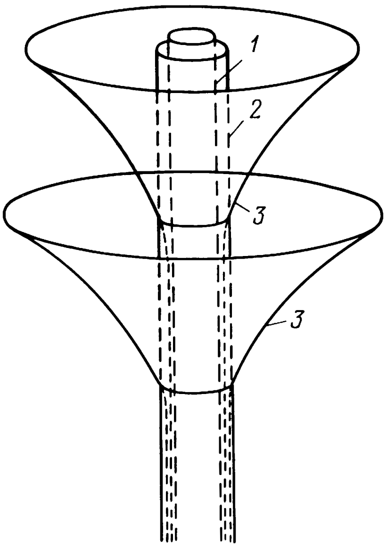


Fig. 69. '3+1' split of spacetime close to the black hole horizon: 1 - black hole event horizon, 2 - 'stretched' horizon, 3 - $t =$ const sections.

As we approach the horizon, these surfaces recede into distant past in parameter V (2.4.11) or in time T of a freely falling observer [T is the time of reference frame (2.4.3) for the Schwarzschild metric or the time in a similar reference frame for the Kerr metric].[5] Hence, as we approach the horizon along this section, it contains the values of the electromagnetic field that correspond to the distant-past history. This fact is of no importance if a stationary problem is considered. But if we are interested in the evolution of the fields, it becomes important and may lead to serious inconveniences. It was therefore suggested that the concept of 'stretched' horizon should be introduced. This horizon is defined as the surface (membrane) that lies in the immediate vicinity and outside of the horizon but is time-like, in conrtrast to the event horizon (see Figure 69). The exact position of the 'stretched' horizon is not specified; it is chosen to suit a concrete problem.

The boundary conditions are fixed in this approach on the 'stretched' horizon. The entire distant past history of the fields at $t =$ const close to the true horizon is then cut off and ignored. A detailed theory of 'stretched' horizon and the relevant bibliography can be found in Price and Thorne (1986), and in the monograph by Thorne *et al.* (1986).

Let us return to the true horizon. As in the preceding section, we introduce 'special' physical conditions. First we assume that the problem is stationary and axisymmetric. The fictitious surface current $\mathfrak{I}^H$ and electric field $\mathbf{E}^H$ at the horizon are then completely poloidal while the magnetic field $\mathbf{B}^H$ is toroidal:

$$\mathfrak{I}_P^H = \frac{I}{2\pi\left(\frac{A^H \sin^2\theta}{\rho_H^2}\right)^{1/2}}, \tag{7.3.10}$$

$$E_P^H = -\frac{2I}{c\left(\frac{A^H \sin^2\theta}{\rho_H^2}\right)^{1/2}}, \tag{7.3.11}$$

$$B_T^H = -\frac{2I}{c\left(\frac{A^H \sin^2\theta}{\rho_H^2}\right)^{1/2}}, \tag{7.3.12}$$

where A^H and ρ_H are the values assumed by A and ρ at the horizon. Furthermore, the poloidal magnetic field measured by nonrotating observers intersects the event horizon at right angles. Recall that the toroidal component diverges at the horizon.

Let us see now what changes arise in the case of a force-free field in the neighborhood of the black hole. First of all we note that the field degeneracy condition (7.2.18) is satisfied on the horizon while condition (7.2.19) is not.

Now the electric field $\mathbf{E}^H$ at the horizon is directly expressible in terms of $\mathbf{B}_\perp$:

$$\mathbf{E}^H = -\frac{1}{c}(\Omega^F - \Omega^H)\mathbf{m} \times \mathbf{B}_\perp. \tag{7.3.13}$$

The following fact is of maximum importance. If the field is force-free, the solutions of the stream equation (7.2.26) without any nonphysical singularity satisfy automatically the boundary conditions at the event horizon. Furthermore, the following 'principle of least action' is found to hold.

The lines of the poloidal magnetic field that intersect the horizon have a distribution that ensures extremal total surface energy $\mathfrak{E}$ of the tangential electromagnetic field at the horizon. The expression for $\mathfrak{E}$ has the form

$$\mathfrak{E} = \frac{1}{8\pi}\int_H [(\mathbf{B}^H)^2 + (\mathbf{E}^H)^2]\, d\Sigma^H, \tag{7.3.14}$$

where the integration is carried out over the horizon.

Now we write Equations (7.3.8) and (7.3.9) for the force-free field in the form

$$dJ = \frac{(\Omega^F - \Omega^H)}{4\pi c}\,\frac{A^H \sin^2\theta}{\rho_H^2}\,(B_\perp)^2\, d\Sigma^H\, dt, \tag{7.3.15}$$

$$\mathrm{d}M = \frac{\Omega^F(\Omega^F - \Omega^H)}{4\pi c^3} \frac{A^H \sin^2\theta}{\rho_H^2} (B_\perp)^2 \,\mathrm{d}\Sigma^H \,\mathrm{d}t. \tag{7.3.16}$$

The angular momentum and energy lost by the black hole are transferred along the lines of force of the poloidal field in the force-free region to those 'regions' where condition (7.2.19) is violated.

Note that if $\Omega^F = 0$ (i.e., if the magnetic lines of force are, say, frozen into plasma far from the black hole and this plasma *does not participate* in the rotation around the hole), then $\mathrm{d}M = 0$, that is, the total mass of the black hole is conserved and $\mathrm{d}J < 0$; in other words, the rotation of the black hole slows down. The entire energy of rotation transforms into the mass of the black hole (the so-called irreducible mass; see Section 8.1) so that nothing escapes.

If the parameters of the black hole are fixed, its angular velocity Ω^F is determined by the boundary condition far from the black hole in the external plasma. The situation in realistic astrophysical conditions will be discussed in Section 7.5.

7.4. Electromagnetic Fields in Vacuum in the Neighborhood of a Black Hole

Before beginning the description of the magnetosphere of a rotating black hole [it is formed via the accretion of magnetized gas (see the next section)], we illustrate the above analysis by the solutions of the following problems on electromagnetic fields in vacuum:

(1) electric charge in the vacuum in the Schwarzschild metric [Copson (1928), Linet (1976), Hanni and Ruffini (1973)];
(2) magnetic field in the vacuum in the Kerr metric, uniform at infinity [Wald (1974b), Thorne and MacDonald (1982), King and Lasota (1977)].

We begin with problem (1). Let a charge q be at rest in the Schwarzschild coordinates at $r = b$, $\theta = 0$. The problem reduces to solving system (7.2.15)–(7.2.17) with δ-function for ρ_ε and $j^P = j^T = 0$. Conditions (7.2.15) and (7.2.16) are satisfied when $\Psi = I = 0$. Expressions (7.2.13) and (7.2.14) then imply that the external magnetic field is absent. External currents also vanish. Hence [see expressions (7.3.2) and (7.3.3)], the surface current at the horizon is also zero, $\mathfrak{I}^H = 0$. As follows from condition (7.3.7), at the horizon $\mathbf{E}_\parallel \to 0$, so that electric lines of force intersect the horizon at right angles. The total flux of $\mathbf{E}$ across the horizon is zero (the black hole is not charged). With these boundary conditions, $\mathfrak{A}_0$ is found from the solution (7.2.17) with $\rho_e = \delta(r{-}b,\theta)q$ and $\omega = 0$; then $\mathbf{E}^P$ is found from (7.2.11) (throughout this section, with the exception of the final formulas, we set $G = 1$, $c = 1$):

$$\mathbf{E}^P = \frac{q}{br^2}\left\{M\left(1 - \frac{b - M + M\cos\theta}{D}\right) + \right.$$

$$+\frac{r[(r-M)(b-M)-M^2\cos\theta]}{D^3}[r-M-(b-M)\cos\theta]\Big\}\mathbf{e}_{\hat{r}}+$$

$$+\frac{q(b-2M)(1-2M/r)^{1/2}\sin\theta}{D^3}\mathbf{e}_{\hat{\theta}}, \qquad (7.4.1)$$

where $\mathbf{e}_{\hat{r}}$ and $\mathbf{e}_{\hat{\theta}}$ are unit vectors along the directions of r and θ, respectively, and

$$D \equiv [(r-M)^2+(b-M)^2-M^2-2(r-M)(b-M)\cos\theta+M^2\cos^2\theta]^{1/2}. \qquad (7.4.2)$$

The pattern of electric lines of force is shown in Figure 70. The charge surface density at the boundary of the black hole follows from (7.3.1):

$$\sigma^H=\frac{q[M(1+\cos^2\theta)-2(b-M)\cos\theta]}{8\pi b[b-M(1+\cos\theta)]^2}. \qquad (7.4.3)$$

Let us bring the charge closer to the horizon ($b \to 2M$). At a distance $r \gg b-2M$ from the horizon, the lines of force become practically radial, and the field strength tends to q/r^2. With the exception of a narrow region close to the horizon, the general picture is almost the same as for a charge placed at the center of the black hole.

Now we will give, without detailed justification, the solution of problem (2).

A rotating black hole is placed in a magnetic field B_0, uniform at infinity. In the Kerr metric, the magnetic field is given by the expression

$$\mathbf{B}=\frac{B_0}{2\rho\sin\theta}\left(\frac{\Delta}{A}\right)^{1/2}\left(\frac{\partial x}{\partial\theta}\frac{\partial}{\partial r}-\frac{\partial x}{\partial r}\frac{\partial}{\partial\theta}\right), \qquad (7.4.4)$$

where $x \equiv (A-4a^2Mr)\cdot\sin^2\theta/\rho^2$.

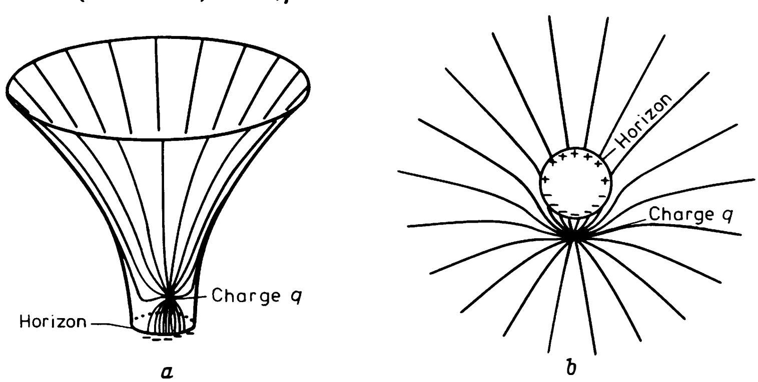


Fig. 70. The electric-field lines of force of a test charge q at rest in the Schwarzschild metric, in a ϕ = const section; (a) lines of force on a curved surface whose geometry coincides with the section ϕ = const of the Schwarzschild metric; (b) the same lines projected on a plane ('top view'). The distribution of the fictitious surface charge σ^H is shown on the horizon. The charge q is assumed to be positive.

The electric field induced by the rotation of the black hole is proportional to a:

$$\mathbf{E} = \frac{-B_0 a A^{1/2}}{\rho^3} \left\{ \Delta^{1/2} \left[\frac{\partial \left(\frac{\rho^2 \Delta}{A} \right)}{\partial r} + \frac{M \sin^2 \theta}{\rho} \times (A - 4a^2 M r) \frac{\partial}{\partial r} \left(\frac{r}{A} \right) \right] \frac{\partial}{\partial r} + \right.$$

$$\left. + \Delta^{-1/2} \left[\frac{\partial \left(\frac{\rho^2 \Delta}{A} \right)}{\partial \theta} + \frac{M r \sin^2 \theta}{\rho^2} (A - 4a^2 M r) \frac{\partial}{\partial \theta} \left(\frac{1}{A} \right) \right] \frac{\partial}{\partial \theta} \right\} . \tag{7.4.5}$$

As in problem (1), $\mathbf{E}^H$, $\mathbf{B}^H$, and $\mathfrak{I}^H$ are absent.[6] Formulas (7.3.8), (7.3.9) imply that the angular momentum J of a black hole and its mass M remain invariant. King and Lasota (1977) demonstrated that the angular momentum changes if the magnetic field is tilted at an angle to the black hole axis. Their result is as follows.

Let a magnetic field $\mathbf{B}$, uniform at infinity, be tilted at an angle to the direction of the angular momentum $\mathbf{J}$. Decompose $\mathbf{J}$ into a component $\mathbf{J}_\parallel$ parallel to the field and a component $\mathbf{J}_\perp$ perpendicular to it. The following formulas give $\mathbf{J}$ as a function of time:

$$\mathbf{J}_\parallel = \text{const}, \tag{7.4.6}$$

$$\mathbf{J}_\perp = \mathbf{J}_\perp(t = 0) \exp(-t/\tau), \tag{7.4.7}$$

where

$$\tau = \frac{3c^5}{2G} \mathbf{M}^{-1} (\mathbf{B})^{-2} = 10^{36} \text{ years} \left(\frac{M}{M_\odot} \right)^{-1} \left(\frac{B}{10^{-5} \text{ Gauss}} \right)^{-2} . \tag{7.4.8}$$

The component $\mathbf{J}_\perp$ of the black hole angular momentum is thus completely lost with time. The rotational energy connected with $\mathbf{J}_\perp$ thus transforms in the static magnetic field into the 'irreducible' mass of the black hole while the component $\mathbf{J}_\parallel$ remains unaltered.

The final state of the black hole corresponds to Hawking's theorem that states that the stationary state must be axially symmetric. Press (1972) points out that if the external magnetic field (or any other field) is not axisymmetric, the black hole finally loses its angular momentum (by Hawking's theorem). If the field $\mathbf{B}$ varies smoothly on a scale much greater than the black hole size, $\mathbf{J}$ can again be decomposed into $\mathbf{J}_\parallel$ and $\mathbf{J}_\perp$ with respect to the field direction in the black hole neighborhood. By the order of the magnitude, the decrease of $\mathbf{J}_\perp$ is again determined by formula (7.4.7), while $\mathbf{J}_\parallel$ decreases according to the formula

$$\frac{d\mathbf{J}_\parallel}{dt} \approx -\frac{\mathbf{J}_\parallel}{\tau} O\left(\frac{r_g}{R} \right), \tag{7.4.9}$$

where R is the scale of nonuniformity of the field.

We have mentioned in the note[6] to p.150 that if $a = a_{max}$, an axisymmetric magnetic field does not intersect the black hole horizon. Bicak (1983, 1985) proved that if a magnetic field B_0, uniform at infinity, is tilted with respect to the rotation axis, then the flux through one half of the horizon [7] for the field component $B_{0\perp}$ is maximal when $a = a_{max}$; it equals

$$\tilde{\Phi}_{max} \approx 2.25 B_{0\perp} \pi M^2. \tag{7.4.10}$$

Finally, consider a *nonrotating* black hole placed in a strong magnetic field B_0, uniform at infinity [Bicak (1983)]. Let the field be so strong that its self-gravitation has to be taken into account. It is then found that for a black hole of fixed mass M, there is a critical field $B_{0,cr}$ at which the flux $\tilde{\Phi}$ through one half of the event horizon is maximal:

$$B_{0,cr} = c^4 G^{-3/2} M^{-1} = 2.5 \times 10^{13} \text{ Gauss} \times \left(\frac{10^6 M_\odot}{M} \right), \tag{7.4.11}$$

$$\tilde{\Phi}_{max,1} = 2\pi G^{1/2} M. \tag{7.4.12}$$

The flux across the horizon cannot be greater than $\tilde{\Phi}_{max,1}$.

About a rotating electrically charged black hole in an external magnitude field, see Dokuchayev (1987).

7.5. Magnetosphere of a Black Hole

The problems discussed in the preceding section illustrate some important properties of electric and magnetic fields in the neighborhood of a black hole. These problems can hardly be used, however, for the description of the actual electrodynamic processes expected to take place in astrophysical conditions. We have already mentioned that the reason for this is the nonvacuum nature of the fields in the vicinity of black holes. The space is always filled with rarefied plasma (so that the fields becomes force-free) or even more complicated situations arise.

Recall that the fields in the neighborhood of rotating magnetized neutron stars (pulsars) cannot be considered as vacuum fields either: A complicated pulsar magnetosphere is formed [Goldreich and Julian (1969), Ruderman and Sutherland (1975), Arons (1983), Beskin *et al.* (1983), Kardashov *et al.* (1984), Gurevich and Istomin (1985)].

By analogy to pulsars, the region of magnetized plasma around a black hole is called the black-hole *magnetosphere*. The complexity and diversity of processes in this region stand in the way of developing a complete magnetosphere theory. In fact, we will lack an acceptably complete theory of a pulsar's magnetosphere, despite the great expenditure of effort and time that has been put into it.

We will not discuss all aspects of the theory of the black-hole magnetosphere here. We will only look at the important side of the electrodynamic processes that occur in the black-hole neighborhood and are black-hole-specific. Correspondingly, we restrict the representation to the simplest model. Only the processes caused by the black hole itself are covered, while those due to, for example, the motion of plasma in the accreted gas disk formed around the black hole are left out.

We choose the stationary axisymmetric model of the magnetosphere of a black hole inducing disc accretion of magnetized gas (the magnetosphere schematically shown in Figure 68). This model was analyzed by Blandford (1976), Blandford and Znajek (1977), MacDonald and Thorne (1982), and Phinney (1983); see also Lovelace (1976), Lovelace *et al.* (1979), Thorne and Blandford (1982), Rees *et al.* (1982).

For the conditions of force-free field existence, (7.2.20) – (7.2.21), to be satisfied in the neighborhood of a black hole, it is necessary to have rarefied plasma with electric currents flowing along magnetic lines of force. Charges for these currents crossing the black hole must be constantly replenished because they sink into the hole and obviously, charges cannot flow back out.[8] Mechanisms of free charge generation thus have to exist in the neighborhood of the black hole. Such mechanisms were analyzed by Blandford and Znajek (1977) and Kardashov *et al.* (1983). We remark, without going into the details, that these mechanisms require a small component of the electric field, parallel to the magnetic field. This component is so small that inequalities (7.2.20) – (7.2.24) are not violated.

Consider a thin tube of the lines of force that pass through the black hole. This tube rotates around the black hole at a constant angular velocity Ω^F (see Section 7.2). As follows from formula (7.3.16), the rotational energy of the black hole is extracted at the rate.

$$\frac{d\mathfrak{E}}{dt} = -c^2\frac{dM}{dt} = \frac{\Omega^F(\Omega^H - \Omega^F)}{4\pi c}\,\frac{A^{H^2}\sin^2\theta}{\rho_H^2}\,B_\perp^2\,d\Sigma^H. \tag{7.5.1}$$

This energy is transferred along magnetic lines of force into region 3 (see Figure 68) where the force-free condition is violated, energy is pumped into accelerated particles, and so forth.

In this region, the particles exert back-reaction on the line of force, owing to their inertia, and, thus, determine Ω^F. If inertia is large, the angular velocity Ω^F is small ($\Omega^F \ll \Omega^H$); in the limit $dt\,\Omega^F \to 0$. The power $d\mathfrak{E}/dt$ of the above 'engine' is the quite low, as follows from (7.5.1). Otherwise (i.e., when the inertia of the particles in the region 3 is low) $\Omega^F \to \Omega^H$, and (7.5.1) again gives low power. The power is maximum when $\Omega^F = \Omega^H/2$.

MacDonald and Thorne (1982) demonstrated that this condition is very likely to be implemented in the described model. Their arguments run as follows. The angular velocity of nonrotating observers far from the black hole can be assumed

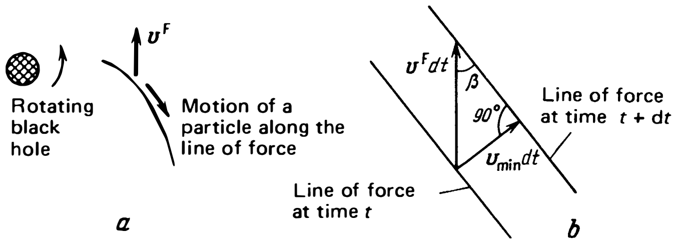


Fig. 71. (a) Scheme of motion of a charged particle along a magnetic field line of force rotating around a black hole. (b) The position of a segment of a magnetic line of force in the plane of the vectors B^P and B^T at the moments t and t+dt. If the velocity vector of a particle that slides outward in the absolute space along a line of force is perpendicular to this line, the velocity of the particle is minimal.

zero, and the velocity $\mathbf{v}^F$ of the points of the lines of force far from the symmetry axis (with respect to nonrotating observers and, hence, in the absolute space, because $\omega = 0$) is much greater than the speed of light:

$$|\mathbf{v}^F| = |\Omega^F \mathbf{m}| \gg c. \tag{7.5.2}$$

Charged particles cannot move at a velocity greater than c. Staying on a line of force, however, and sliding along it outward (see Figure 71a), they can have velocities less than $|\mathbf{v}^F|$. Figure 71b shows a segment of a line of force in the plane defined by the vectors $\mathbf{B}^P$ and $\mathbf{B}^T$ There is an optimal velocity of sliding along the line of force (which, in turn, moves at a velocity $\mathbf{v}^F$), such that the total velocity of the particle in the absolute space is minimal (this is clear from the Figure 71b). The angle β is the angle between the direction of the line of force and the direction of the ϕ coordinate; hence, β is found from the relation

$$\sin\beta = \frac{|\mathbf{B}^P|}{\sqrt{(\mathbf{B}^P)^2 + (\mathbf{B}^T)^2}}. \tag{7.5.3}$$

Using (7.5.3), we find $|\mathbf{v}_{min}|$:

$$|\mathbf{v}_{min}| = \frac{|\mathbf{v}|^F}{\sqrt{1 + (\mathbf{B}^T)^2/(\mathbf{B}^P)^2}}. \tag{7.5.4}$$

It is found that the condition $|\mathbf{v}_{min}| \approx c$ is equivalent to the maximal energy condition $\Omega^F \approx \Omega^H/2$. Indeed, let us write for $|\mathbf{v}^F|$, $|\mathbf{B}^T|$ and $|\mathbf{B}^P|$ the following expression valid *far from both the black hole and the symmetry axis.*

For $|\mathbf{v}^F|$ we find

$$|\mathbf{v}^F| \approx \Omega^F |\mathbf{m}|. \tag{7.5.5}$$

Making use of formulas (7.2.14), (7.3.11), (7.3.13), and definition (7.2.2), we obtain

$$|\mathbf{B}^T| \approx \frac{(\Omega^H - \Omega^F)\Psi}{\pi c|\mathbf{m}|}. \tag{7.5.6}$$

Finally, formula (7.2.13) and relation $|\nabla\Psi| \approx 2\Psi/|\mathbf{m}|$ yield

$$|\mathbf{B}^P| \approx \frac{\Psi}{\pi|\mathbf{m}|^2}. \tag{7.5.7}$$

Substituting (7.5.5) – (7.5.7) into (7.5.4), we find

$$|\mathbf{v}_{\min}| \approx \frac{c\Omega^F}{\Omega^H - \Omega^F}. \tag{7.5.8}$$

The last formula implies that $|\mathbf{v}_{\min}| \approx c$ when $\Omega^F = \Omega^H/2$. If $\Omega^F \ll \Omega^H/2$ and $|\mathbf{v}_{\min}| \ll c$, then the inertia of particles in the region 3 is small and Ω^F increases until the velocity $|\mathbf{v}_{\min}|$ grows close to c. If $|\mathbf{v}_{\min}| \gg c$, then particles cannot stay long on the lines of force and their back-reaction on the field reduces Ω^F until we obtain $|\mathbf{v}_{\min}| \approx c$.[9]

These are the reasons why it is likely that $\Omega^F \approx \Omega^H/2$ and the rate of extraction of rotational energy from the black hole (7.5.1) is almost optimal.

By the order of magnitude, the power of the 'electric engine' outlined above is

$$P \approx \left(10^{40}\,\frac{\text{erg}}{s}\right)\left(\frac{M}{10^6 M_\odot}\right)^2\left(\frac{a}{a_{\max}}\right)^2\left(\frac{B}{10^4\ \text{Gauss}}\right)^2. \tag{7.5.9}$$

Here B is the magnetic field strength in the neighborhood of the black hole. Sometimes this electric engine is described in terms of electrical engineering [Blandford (1979), Znajek (1978), Damour (1978), MacDonald and Thorne (1972), Thorne and Blandford (1982)]. We will give the expression for quantities at the black hole horizon using this notation.

Equipotential curves at the horizon are the lines of constant θ, since the field $\mathbf{E}^H$ is meridional [see (7.3.11)]. Hence, the potential difference between two equipotential lines (marked by 1 and 2) are [see also (7.3.13)].

$$\Delta U^H = \int_1^2 E^H\,\mathrm{d}l = (\Omega^H - \Omega^F)\frac{\Delta\Psi}{2\pi c} \approx (10^{18}\text{V})\left(\frac{M}{10^6 M_\odot}\right)\left(\frac{B}{10^4\text{G}}\right). \tag{7.5.10}$$

where $\mathrm{d}l$ is an element of distance along a meridian on the black hole surface, and $\Delta\Psi$ is the difference between the values of Ψ on the equipotential 1 and 2. The approximate equality in (7.5.10) is written for the conditions $\Omega^F \approx \Omega^H/2$, maximal Ω^H, and the equipotentials 2 and 1 corresponding to the equatorial and polar regions, respectively.

On the other hand, ΔU^H can be written in terms of the surface current $\mathfrak{I}^H$ and resistance:

$$\Delta U^H = R^H \mathfrak{F}^H \Delta l\,, \tag{7.5.11}$$

where Δl is the distance along the meridian between the equipotentials 2 and 1.

Substituting expression (7.3.10) for $\mathfrak{F}^H$, we obtain

$$\Delta U^H = \frac{I R^H |\Delta l|}{2\pi \left(\frac{A^H \sin^2 \theta}{\rho_H^2} \right)^{1/2}} = I\, \Delta Z^H, \tag{7.5.12}$$

where

$$\Delta Z^H \equiv \frac{R^H |\Delta l|}{2\pi \left(\frac{A^H \sin^2 \theta}{\rho_H^2} \right)^{1/2}} \tag{7.5.13}$$

is the total resistance between the equipotential lines 2 and 1 (if the equipotentials 2 and 1 correspond to the equator and to $\theta \approx \pi/4$, the integration of (7.5.13) yields $\Delta Z^H \approx$ 30 ohm).

Formulas (7.5.10) and (7.5.13) permit the conclusion that in this model the rotating black hole acts as a battery with e.m.f. in the order of

$$(10^{18} V) \cdot \left(\frac{M}{10^6 M_\odot} \right) \cdot \left(\frac{B}{10^4\ \text{Gauss}} \right)$$

and internal resistance of about 30 ohm.

This mechanism (and a number of its versions) has been employed in numerous papers for the explanation of the activity of the nuclei of galaxies and quasars [see, for example, Blandford (1976), Blandford and Znajek (1977), Blandford (1979), Ruffini and Wilson (1975), Lovelace *et al.* (1979), Blandford and Rees (1978), Kardashov *et al.* (1983), Rees (1982), Novikov and Shtern (1986)].

Chapter 8

Physical Effects in the Gravitational Field of a Black Hole

8.1. Extraction of Energy from a Black Hole. Superradiance

In this chapter we continue the discussion of the effects of interaction between classical particles and fields and black holes.[1] Let us begin by analyzing the efficiency of energy extraction from black holes. Recall that although a black hole is by definition a region of forbidden escape for any body and light rays, situations are possible in which energy can be extracted via certain physical processes. We will see later that this energy is extracted from the field connected with the black hole and surrounding it. Thus, extraction is possible if the black body rotates or is charged. Among such processes we already know the Penrose process (see Section 6.2) and the electrodynamic processes discussed in the preceding chapter. In this section, we establish some general restrictions on the possible efficiency of this kind of process.

Consider the efficiency of the Penrose process (see Figure 64). Let $\varepsilon_i = -p_i^{\mu}\xi_{(t)\mu}$ be the energy and $j_i = p_i^{\mu}\xi_{(\phi)\mu}$ the angular momentum of a particle of type i with momentum p_i^{μ}, that moves in the gravitational field of a Kerr black hole ($i = 0$ denotes a falling particle decaying in the ergosphere, $i = 1$ a particle ejected to infinity, and $i = 2$ a particle absorbed into the black hole). Note now that at the event horizon the vector

$$l^{\mu} = \xi^{\mu}_{(t)} + \Omega^H \xi^{\mu}_{(\phi)}, \tag{8.1.1}$$

where Ω^H is the angular velocity of the black hole, is a null vector tangent to the generators of the horizon. Since p_2^{μ} is a time-like vector and l^{μ} is future-directed, we have

$$0 \geq l^{\mu} p_{2\mu} = -\varepsilon_2 + \Omega^H j_2. \tag{8.1.2}$$

Therefore, a particle sinking into the black hole obeys the inequality

$$j_2 \leq \varepsilon_2/\Omega^H. \tag{8.1.3}$$

Thus, if the outgoing particle has a higher energy than the falling one ($\varepsilon_1 - \varepsilon_0 = -\varepsilon_2 > 0$), a similar relation holds for angular momenta:

$$j_1 - j_0 = -j_2 \geq -\varepsilon_2/\Omega^H \geq 0. \tag{8.1.4}$$

The parameters M and J of the black hole change when a particle is absorbed,

$$\delta M = \varepsilon_2, \ \ \delta J = j_2, \tag{8.1.5}$$

with condition (8.1.3) signifying that

$$\delta M \geq \Omega^H \, \delta J. \tag{8.1.6}$$

The physical processes, resulting in such changes δM and δJ of black hole parameters that they obey the relation

$$\delta M / \Omega^H - \delta J = 0, \tag{8.1.7}$$

are said to be *reversible.*

The differential equation (8.1.7) relating the changes in M and J in a reversible process can be integrated [Christodoulou (1970)]. Note that the total differential of the function

$$\tilde{A} = M^2 + \sqrt{M^4 - J^2} \tag{8.1.8}$$

can be written in the form

$$\delta \tilde{A} = \frac{J}{\sqrt{M^4 - J^2}} \left(\frac{\delta M}{\Omega^H} - \delta J \right). \tag{8.1.9}$$

Here

$$\Omega^H = \frac{a}{r_+^2 + a^2} = \frac{a}{2Mr_+} = \frac{J}{2(M^2 + \sqrt{M^4 - J^2})}.$$

Relations (8.1.6) and (8.1.9) show that the following inequality holds for the above-discussed processes involving a particle falling into a black hole:

$$\delta \tilde{A} \geq 0; \tag{8.1.10}$$

the equality holds if and only if the process is reversible. The quantity

$$M_{\rm ir} = (\tilde{A}/2)^{1/2} \tag{8.1.11}$$

is known as the *irreducible mass* of the black hole [Christodoulou (1970)]. Equations (8.1.8) and (8.1.11) give

$$M^2 = M_{\rm ir}^2 + \frac{1}{4}\frac{J^2}{M_{\rm ir}^2} \geq M_{\rm ir}^2. \tag{8.1.12}$$

This relation implies that the Penrose process cannot make the initial mass M less

than M_{ir} and, hence, the maximum possible energy gain in this process is $\Delta\mathcal{E} = \Delta Mc^2$, where

$$\Delta M = M_0 - M_{ir}(M_0, J_0), \tag{8.1.13}$$

and M_0 and J_0 are the initial mass and angular momentum of the black hole; $M_{ir}(M_0, J_0)$ is the corresponding to them irreducible mass.

Simple arguments show that for a given initial mass M_0, the maximal value of ΔM,

$$\Delta M_{max} = (1 - 1/\sqrt{2})M_0 \approx 0.29\, M_0, \tag{8.1.14}$$

is reached for the extremal black hole with $J_0 = M_0^2$.

It is easily shown that the quantity $\tilde{A}$ differs from the area A of the Kerr black hole only by the numerical coefficient

$$A \equiv 4\pi(r_+^2 + a^2) = 8\pi\tilde{A}. \tag{8.1.15}$$

In view of this, condition (8.1.10) that signifies nondecreasing black hole area in the processes analyzed above, is in fact a particular case of the general Hawking's theorem (Section 5.4).

Hawking's theorem makes it possible to draw a number of general conclusions on the processes involving black holes. First of all, inequality (8.1.6) can easily be extended to charged black holes and to processes involving charged particles. It is done simply by making use of expression (8.1.15) where r_+ for a charged black hole is

$$r_+ = M + \sqrt{M^2 - a^2 - Q^2}. \tag{8.1.16}$$

The condition $\delta A \geq 0$ then yields

$$\delta M \geq \Omega^H\, \delta J + \Phi^H\, \delta Q, \tag{8.1.17}$$

where δJ and δQ are the changes in the angular momentum and electric charge of the black hole, and

$$\Phi^H = Qr_+/(r_+^2 + a^2) \tag{8.1.18}$$

is its electric potential.

Processes for which equality is satisfied in (8.1.17), which generalizes (8.1.16), are also said to be reversible. The property common to all reversible processes is that the area of black holes does not increase.

It should be emphasized that δJ in (8.1.17) is the total change in the angular momentum of a black hole. Whether this change is caused by the angular momentum of the incoming particle (corresponding to its orbital motion), or by the inherent angular momentum (spin), is irrelevant. The generalized inequality (8.1.17) applied to the latter case serves to show, among other things, that additional gravitational spin-spin interaction exists between the spin of the particle and the

angular momentum of a rotating black hole (Hawking (1972a), Wald (1972), Bekenstein (1973b)].

As an illustration, consider the simplest case of a particle of spin s, charge e, and energy ε falling on a black hole and moving exactly along the symmetry axis. If this particle sinks into the black hole, we make use of conservation laws and obtain

$$\delta Q = e, \quad \delta J = \sigma s, \quad \delta M \leq \varepsilon. \tag{8.1.19}$$

Here $\sigma = 1$ if the spin points in the direction of rotation of the black hole, and $\sigma = -1$ otherwise. The inequality in the last of relations (8.1.19) is possible because energy may be partly radiated away. Relations (8.1.17) and (8.1.19) show that a particle with nonzero spin can fall into a black hole only if its energy ε exceeds the quantity $\sigma s \Omega^H + e\Phi^H$. The second term, $e\Phi^H$, carries the meaning of standard electrostatic energy of repulsion. The first term describes repulsion ($\sigma = 1$) or attraction ($\sigma = -1$) due to the spin-spin interaction in gravitation theory. [This interaction exists between any two rotating bodies; for a detailed derivation of the expression for this force, and the description of the analogy of the gravitational spin-spin interaction and electromagnetic interaction between magnetic dipoles, see Wald (1972).]

It is logical to expect that under certain conditions a wave incident on a rotating black hole may also be enhanced since the motion of particles in the geometrical optics is directly related to the propagation of wave packets. Let us ascertain (using Hawking's theorem) that this process is indeed possible, and derive the conditions under which it takes place.

The Kerr–Newman metric describing the geometry of charged black holes is stationary and axially symmetric; hence, expansion in eigenfunctions of the operators $\xi^\mu_{(t)}\partial_\mu \equiv \partial_t$ and $\xi^\mu_{(\phi)}\partial_\mu \equiv \partial_\phi$ is convenient in describing wave propagation in the background of this metric.

Consider the behavior of the field mode ϕ_A with quantum numbers ω, m; the temporal and angular dependence of the mode is

$$\phi_A \sim f_A(r, \theta) \exp(-i\omega t + im\phi). \tag{8.1.20}$$

The field ϕ_A can describe scalar, electromagnetic, and gravitational waves [2] (or other boson fields whose quanta may have, for instance, mass μ and charge e). Far from the black hole, solution (8.1.20) describes the ensemble of quanta, each having energy $\hbar\,\omega$, a ϕ-component of angular momentum $\hbar\, m$, and possibly an electric charge e. Hence, the ratio of the flux of the ϕ-component of angular momentum and of the electric charge across a large-radius sphere surrounding the black hole to the flux of energy across this sphere are equal to m/ω and $e/\hbar\,\omega$, respectively. (This can be proved rigorously by using explicit expressions for the energy momentum tensor and current corresponding to the field ϕ_A.) Making use of the energy and angular momentum conservation laws that reflect the symmetry of the problem in hand, and electric charge conservation, one can show that the interaction of the wave ϕ_A with the black hole changes the black hole mass, angular momentum, and charge by δM, δJ and δQ related as follows

$$\delta J = \frac{m}{\omega}\,\delta M, \;\; \delta Q = \frac{e}{\hbar\omega}\,\delta M. \tag{8.1.21}$$

Using now inequality (8.1.17) implied by Hawking's theorem, we obtain

$$\delta M\left(1 - \frac{m\Omega^H}{\omega} - \frac{e\Phi^H}{\hbar\omega}\right) \geq 0. \tag{8.1.22}$$

Thus scattering of the modes satisfying the condition

$$\hbar\,\omega < \hbar\, m\Omega^H + e\Phi^H, \tag{8.1.23}$$

results in reducing the mass of black hole. If this condition is satisfied, the scatterred wave possesses an energy greater than the incident wave, that is, the incident wave is amplified [Zel'dovich (1971, 1972), Misner (1972), Starobinsky (1973), Starobinsky and Churilov (1973), Unruh (1974)]. This effect is known as *superradiance*.

The possibility of wave amplification by rotating black holes was first noticed by Zel'dovich (1971, 1972) who employed an analogy of such black holes to rotating absorbing bodies. As for the latter bodies, the amplification effect described by Zel'dovich is related to a certain extent to the Vavilov–Cherenkov effect. Indeed, consider in the ordinary flat space a cylindrical wave incident on a cylinder of radius R, rotating at an angular velocity Ω around an axis coinciding with the z axis. The corresponding solution ϕ_A has the form

$$\phi_A = f_A(\rho)\exp[i(m\phi - \omega t)]. \tag{8.1.24}$$

On the cylinder surface ($\rho = R$) this field corresponds to a perturbation propagating at a phase velocity $d\phi/dt = \omega/m$. If the velocity ΩR at which matter moves on the surface of a dielectric or conducting cylinder is greater than the linear velocity $R\omega/m$ at which the phase of the incident wave moves on the cylinder surface, absorption of the wave is replaced with its amplification. The corresponding condition is

$$\omega < \Omega m. \tag{8.1.25}$$

A detailed discussion of the relevant aspects can be found in Bolotovsky and Stolyarov (1980).

Note that the amplification condition (8.1.23) is universal and independent of the spin of the field. For particles with nonzero spin, m corresponds to the quantum number of total (orbital and spin) angular momentum. The wave amplification factor essentially depends on the spin of the field. The maximum increase in the energy of electromagnetic waves is 4.4%, but for the gravitational wave it is 138% [Starobinsky and Churilov (1973)]. Under certain conditions, this amplification is possible for the gravitational radiation of a particle moving close to a rotating black hole. If the particle gained the same amount of energy that it emited to infinity, it could orbit the black hole without ever falling on it: This particle would be a sort of catalyst for extracting energy from a black hole. Such orbits are called '*floating*' [Misner (1972), Press and Teukolsky (1972)]. Alas, Detweiler was able to show

(1978) that regardless of the angular velocity values of the black hole and test particle, this process would not be possible.

The following curious effect is connected with superradiance [Damour *et al.* (1976), Ternov *et al.* (1978), Zouros and Eardley (1979), Detweiler (1980), Gal'tsov (1986)]. Let a wave packet of a massive scalar field revolve on a circular orbit around a rotating black hole, and let the boundary energy on this orbit be such that massive particles in this packet cannot be radiated to infinity. However, these particles can move across the event horizon. If the frequency of quanta falling into the black hole satisfies the condition of superradiance, the fall is accompanied with a more intensive outward radiation. The particles of this radiation have the same quantum numbers as the particles in the packet; hence, they cannot escape to infinity and have to accumulate close to the orbit of the packet. This, finally, results in an instability. Ternov *et al.* (1978) and Detweiler (1980) proved that this instability holds for the scalar field of mass μ such that $\mu M/m_{pl}^2 \ll 1$; the characteristic time of the evolution of instability is

$$\tau = 24(a/M)^{-1}(m_{pl}^2/\mu M)^8(m_{pl}/\mu)t_{pl}, \tag{8.1.26}$$

where

$$m_{pl} = (\hbar\, c/G)^{1/2} \approx 2.18\times 10^{-5}\,\text{g},$$

$$t_{pl} = (\hbar G/c^5)^{1/2} \approx 5.39\times 10^{-43}\,\text{s} \tag{8.1.27}$$

are the Planck mass and Planck time, respectively. A more detailed discussion of the behavior of massive fields in the spacetime of a rotating black hole can found in Gal'tsov (1986) and Gal'tsov *et al.* (1988). No such instability develops with massless fields [Detweiler and Ipser (1973), Press and Teukolsky (1973), Teukolsky and Press (1974)].

It should be mentioned that even though the processes of energy loss by the black hole described earlier in this section (Penrose process and superradiance) are extremely and principally important for black hole physics, they could hardly produce appreciable observable effects in actual astrophysical conditions [Mashhoon (1973), Wald (1974c), Kovetz and Piran (1975a,b)]. Analogues of the Penrose process, in which the decay of a particle is replaced with a collision of two particles, may have more attractive implications. The collision produces two new particles one of which is ejected to infinity [Piran *et al.* (1975)]. One version of this effect is the Compton scattering of a freely falling photon on an electron with a large angular momentum, moving in the ergosphere [Piran and Shaham (1977)].

8.2. Global Structure of the Field of a Test Charge in the Spacetime of an Eternal Black Hole

An approach which is natural for studying the processes outside a black hole was described in the preceding chapter. It consists of reducing the physical problem in the entire spacetime to a problem in the exterior region, by fixing specially chosen boundary conditions on the black hole surface. This approach has certain advantages but is incomplete in principle because it ignores the entire realm of phenomena occurring inside the black hole. It should be emphasized that an analysis of such phenomena, including, for example, a detailed description of specific features of processes in those regions of spacetime where classical theory finds singularities, is an important problem of black-hole physics. Quantum specifics of physical interactions manifest themselves to the full in these regions of large spacetime curvature. The general aspects of the structure of spacetime inside a black hole will be discussed in Chapter 12, after the theory of quantum effects in black holes has been presented. In this section we take up a particular problem of the structure of physical fields created by test charges in the total spacetime of a black hole. The black hole metric is considered to be specified and the effect of the field of the test charge on the metric is neglected.

Let a point-like test charge of an electric (e) or scalar massless (g) field be at rest outside a charged eternal black hole.[3] The world line of this charge is denoted by γ in Figure 72 which gives the Penrose diagram for the spacetime in question. In coordinates t, r that cover the region I shown in this figure, the Reissner–Nordström metric, describing the gravitational field of the charged black hole, is of the form

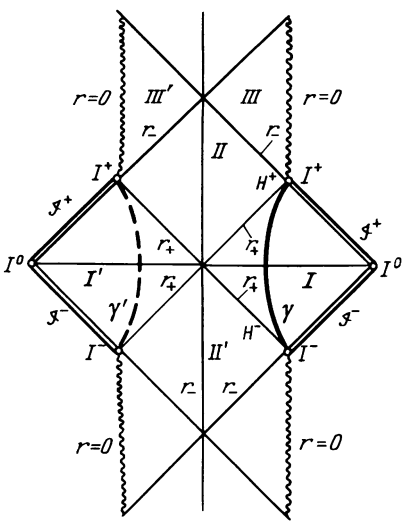


Fig. 72. Penrose diagram for the total spacetime of the Reissner–Nordström black hole.

$$ds^2 = -F(r)\,dt^2 + F^{-1}(r)\,dr^2 + r^2\,d\omega^2,$$

$$F = 1 - 2M/r + Q^2/r^2,\ d\omega^2 = d\theta^2 + \sin^2\theta\,d\phi^2, \tag{8.2.1}$$

where M is the mass and Q is the electric charge of the black hole.

The electric (A_μ) and scalar (ϕ) fields created by the test charge at rest at a point r_0, θ_0, ϕ_0 are found as the solutions of the equations

$$F^{\mu\nu}{}_{;\nu} = 4\pi j^\mu,\ F_{\mu\nu} = 2A_{[\nu,\mu]}, \tag{8.2.2a}$$

$$\Box\phi = -4\pi\rho, \tag{8.2.2b}$$

where

$$j^\mu(x) = e\,\delta^\mu_0 \frac{\delta^3(\mathbf{x} - \mathbf{x}_0)}{\sqrt{-g(\mathbf{x})}},\ \rho(x) = g\frac{\sqrt{-g_{tt}(\mathbf{x})}}{\sqrt{-g(\mathbf{x})}}\,\delta^3(\mathbf{x} - \mathbf{x}_0),$$

$$\delta^3(\mathbf{x} - \mathbf{x}_0) \equiv \delta(r - r_0)\,\delta(\theta - \theta_0)\,\delta(\phi - \phi_0). \tag{8.2.3}$$

The solutions of these equations outside the black hole (in region I) were obtained by Linet (1976) and Lèaute and Linet (1976) who found and corrected a minor inaccuracy in the formula derived earlier (for $Q \equiv 0$) by Copson (1928). These solutions are[4] [cf. (7.4.1)]

$$A_\mu(x) = \delta^0_\mu\, eG_{\text{em}}(x, x_0),\ G_{\text{em}}(x, x_0) = -\frac{1}{rr_0}\left(M + \frac{\Pi}{R}\right), \tag{8.2.4a}$$

$$\phi(x) = gF(r_0)^{1/2}G_{\text{sc}}(x, x_0),\ G_{\text{sc}}(x, x_0) = \frac{1}{R}, \tag{8.2.4b}$$

$$R^2 \equiv R^2(\mathbf{x}, \mathbf{x}_0)$$
$$= (r - M)^2 + (r_0 - M)^2 - 2(r - M)(r_0 - M)\lambda - (M^2 - Q^2)(1 - \lambda^2),$$

$$\Pi \equiv \Pi(\mathbf{x}, \mathbf{x}_0) = (r - M)(r_0 - M) - (M^2 - Q^2)\lambda,$$

$$\lambda = \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\phi - \phi_0). \tag{8.2.5}$$

This solution can be extended from region I to the entire spacetime, using the method of analytic continuation. Accordingly, we introduce in region I, II', I', II' the coordinates u, v, θ, ϕ that are regular in these regions while in region I they are related to the coordinates r, t, θ, ϕ by the formulas

$$u = -\exp\left[-\frac{r_+ - r_-}{2r_+^2}(t - r^*)\right],\quad v = \exp\left[\frac{r_+ + r_-}{2r_+^2}(t + r^*)\right],$$

$$r^* = r + \frac{r_+^2}{r_+ - r_-} \ln\left|\frac{r}{r_+} - 1\right| - \frac{r_-^2}{r_+ - r_-} \ln\left|\frac{r}{r_-} - 1\right|,$$

$$r_\pm \equiv M \pm \sqrt{M^2 - Q^2}, \quad \theta = \theta, \quad \phi = \phi. \tag{8.2.6}$$

If $-\infty < u, v < \infty$, these coordinates cover regions I, II, I', II'. The analytic continuation of the Reissner–Nordström metric (8.2.1) then takes the form

$$ds^2 = -2B\, du\, dv + r^2\, d\omega^2, \tag{8.2.7}$$

where

$$B = \frac{2r_+^4}{r^2} \frac{(r - r_+)(r - r_-)}{(r_+ - r_-)^2} \exp\left(-r^* \frac{r_+ - r_-}{r_+^2}\right),$$

and $r = r(u, v)$ is determined from the equation

$$uv = \left(1 - \frac{r}{r_+}\right) \exp\left[\left(r - \frac{r_-^2}{r_+ - r_-} \ln\left|\frac{r}{r_-} - 1\right|\right)\left(\frac{r_+ - r_-}{r_+^2}\right)\right]. \tag{8.2.8}$$

In these coordinates, the nonzero components of the analytically continued solutions of (8.2.2) are

$$A \equiv A_\mu dx^\mu = -\frac{e}{rr_0}\left(M + \frac{\Pi}{R}\right) \frac{r_+^2}{r_+ - r_-}\left(\frac{dv}{v} - \frac{du}{u}\right), \tag{8.2.9}$$

$$F^{uv} = \frac{e}{Br^2 r_0}\left\{M + \frac{\Pi}{R} - \frac{r}{R^3}[r_0 - M - (r - M)\cos\theta](r_0^2 - 2Mr_0 + Q^2)\right\},$$

$$F^{v\theta} = \frac{e}{2rr_0} \frac{\sin\theta}{R^3} \frac{r_+ - r_-}{r_+^2} (r_0^2 - 2Mr_0 + Q^2) v, \tag{8.2.10}$$

$$F^{u\theta} = -\frac{e}{2rr_0} \frac{\sin\theta}{R^3} \frac{r_+ - r_-}{r_+^2} (r_0^2 - 2Mr_0 + Q^2) u,$$

$$\phi = gF(r_0)^{1/2} \frac{1}{R}. \tag{8.2.11}$$

In these formulas, r is assumed to be a function of u and v, defined by (8.2.8). It is not difficult to see that this solution is invariant under the transformation $u \to -u$, $v \to -v$ that maps region I into I'. Hence, it has, in addition to a singularity corresponding to the world line γ of the charge, a singularity on the line γ' corresponding to an additional charge $-e$ (or $-g$ in the case of a scalar fields) in region I'. As a result, expressions (8.2.9)–(8.2.11) do not give the solution of the formulated problem of finding the field produced by a single charge.

The sought solution can be obtained if we take into account that it would be natural to chose zero field in regions I' and II' that lie outside the region of influence of the test charges.[5] This solution can be written in the following form [Zel'nikov and Frolov (1982, 1983c), Demyansky and Novikov (1982)]:

$$\tilde{F}_{\mu\nu} = F^{\text{reg}}_{\mu\nu} + F^{\text{sing}}_{\mu\nu}, \quad F^{\text{reg}}_{\mu\nu} = F_{\mu\nu}\theta(v), \quad F^{\text{sing}}_{\mu\nu} = \Psi_{\mu\nu}\delta(v), \tag{8.2.12a}$$

$$\tilde{\phi} = \phi^{\text{reg}} + \phi^{\text{sing}}, \quad \phi^{\text{reg}} = \phi\theta(v), \quad \phi^{\text{sing}} = \Psi\delta(v), \tag{8.2.12b}$$

where $F_{\mu\nu}$ and ϕ are solutions (8.2.10) and (8.2.11). Singular terms $F^{\text{sing}}_{\mu\nu}$ and ϕ^{sing} ensure that homogeneous field equations are satisfied on the surface $v = 0$. Substituting (8.2.12a) into Maxwell's equations (8.2.2a), we obtain

$$\Psi^{\mu v} = 0, \quad \frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}\ \Psi^{\mu\nu}) = -F^{\mu v}|_{v=0}. \tag{8.2.13}$$

The solution of these equation, bounded on H^-, takes the form

$$\Psi_{\mu\nu} = -2e\,\delta^{v}_{[\mu}\delta^{\theta}_{\nu]}\sin\theta\,\frac{\sqrt{M^2-Q^2}}{r_0}\,\frac{M+\sqrt{M^2-Q^2}}{r_0 - M - \sqrt{M^2-Q^2}\cos\theta}. \tag{8.2.14}$$

Likewise, substitute (8.2.12b) into the equation for scalar field (8.2.2b); this yields

$$(1-\lambda^2)\frac{\partial^2\Psi}{\partial\lambda^2} - 2\lambda\frac{\partial\Psi}{\partial\lambda} - \frac{r_+ - r_-}{r_+}\Psi = 0, \tag{8.2.15}$$
$$\frac{\partial\Psi}{\partial u} = 0, \quad \lambda = \cos\theta.$$

If the black hole is not extremal ($Q < M$), the only solution (8.2.15) bounded on H^- is $\Psi = 0$. If the black hole is extremal ($Q = M$), another solution is Ψ=const although the value of $\Psi|_{H^-}$ is not determined by the external scalar field and thus has no relation to the charge g. Hence, the solution of the formulated problem in the regions I, II, I', II' is given by relations (8.2.12a), (8.2.12b), and (8.2.13) for $\Psi = 0$.

The analytical continuation readily yields A_μ and ϕ in region III'. In general, it is not possible to extend the solution in a single-valued manner beyond the Cauchy horizon (e.g., in region III) because in this region the solution depends on conditions that must be additionally specified. In the limit of $Q = 0$, the obtained

solution describes the fields produced by point-like sources in the spacetime of an eternal Schwarzschild black hole whose metric in the coordinates u, v, θ, ϕ is

$$ds^2 = -2B\, du\, dv + r^2\, d\omega^2, \tag{8.2.16}$$

where

$$B = \frac{2r_g^3}{r}\exp(-r/r_g),\; r_g = 2\,M,$$

$$uv = \left(1 - \frac{r}{r_g}\right)\exp(r/r_g). \tag{8.2.17}$$

(These coordinates are related to the coordinates $\tilde{u}$, $\tilde{v}$ [see (2.7.12)] by the relations $u = \tilde{v} - \tilde{u}$, $v = \tilde{v} + \tilde{u}$.)

A δ-type singularity in the solution obtained above for the electromagnetic field appears because the problem was so formulated that the charge was assumed to be permanently at rest in the neighborhood of the black hole. A similar singularity appears also in the complete solution describing a massive vector field produced by a source outside the black hole [Frolov (1978a, 1986)].[6] The singularities at $v = 0$, described above are smoothed if one chooses the solutions describing the case of a charge introduced into the field of the black hole.

It is interesting to note that the complete solution derived above for the field of a test charge in the black hole's spacetime is closely related to the solution for the field of a uniformly accelerated charge in flat spacetime [Zel'nikov and Frolov (1982, 1983c)]. In order to find this relation, we notice that if the parameter M in metric (8.2.16) describing the gravitational field of a black hole tends to infinity, then the effect of curvature in the finite region of spacetime close to the event horizon decreases infinitely; hence, the gravitational field in this region can be regarded as uniform to a high degree of accuracy. The transition to the limit of uniform field in metric (8.2.16) is carried out formally as follows. Introduced coordinates U, V, X, Y related to u, v, θ, ϕ by the formulas

$$U = 4\,M\,e^{-1/2}u,\quad V = 4\,M\,e^{-1/2}v,$$

$$X^2 + Y^2 = 16\,M^2\tan^2(\theta/2),\quad Y/X = \tan\phi. \tag{8.2.18}$$

In these coordinates, metric (8.2.16) becomes

$$ds^2 = -\frac{2\,M}{r}\,e^{1-r/2M}\,dU\,dV + \left(\frac{r}{2M}\right)^2 \frac{dX^2 + dY^2}{\left(1 + \dfrac{X^2 + Y^2}{16\,M^2}\right)}, \tag{8.2.19}$$

where r is related to U and V by the expression

$$UV = 16\,M^2\left(1 - \frac{r}{2M}\right)\,\exp\left(\frac{r}{2M} - 1\right). \tag{8.2.20}$$

If M now tends to infinity with the values of the coordinates U, V, X, Y fixed, (8.2.19) transforms into the metric of flat space,

$$\begin{aligned} ds^2 &= -dU\,dV + dX^2 + dY^2 = -dT^2 + dZ^2 + dX^2 + dY^2 \\ &= -dT^2 + dZ^2 + d\rho^2 + \rho^2\,d\phi^2. \end{aligned} \tag{8.2.21}$$

Here

$$T = \frac{1}{2}(U + V), \quad Z = -\frac{1}{2}(U - V), \quad \rho^2 = X^2 + Y^2, \quad \tan\phi = \frac{Y}{X}. \tag{8.2.22}$$

The equation $r = r_0$ of motion of a test charge then takes the form

$$U_0V_0 \equiv -Z_0^2 + T^2 = -w^{-2}, \quad X_0 = Y_0 = 0, \tag{8.2.23}$$

where w is the modulus of the four-acceleration of motion of the charge. In the limit $M \to \infty$, the horizon surfaces $H^{\pm}$ turn into null hyperplanes described by the equation $UV = 0$ [the 'horizons' of the Rindler space (1966)]. The invariant distance to the horizon for a particle with four-acceleration w then tends to a finite value w^{-1}.

In the limiting transition described above, expressions (8.2.12) for $Q = 0$ take the form

$$\begin{aligned} \tilde{F}_{UV} &= \frac{2e}{w^2}\frac{(\rho^2 + \xi + w^{-2})}{S^3}\,\theta(V), \\ \tilde{F}_{U\rho} &= -\frac{4e\rho V}{w^2S^3}\,\theta(V), \\ \tilde{F}_{V\rho} &= \frac{4e\rho U}{w^2S^3}\,\theta(V) - \frac{2e\rho}{\rho^2 + w^{-2}}\,\delta(V), \end{aligned} \tag{8.2.24}$$

$$\phi = \frac{g}{w}\frac{2\theta(V)}{S}, \tag{8.2.25}$$

where

$$S = [(\rho^2 - \xi + w^{-2})^2 + 4\xi w^{-2}]^{1/2}, \quad \xi = UV. \tag{8.2.26}$$

Expressions (8.2.24) for $\tilde{F}_{\mu\nu}$ can be derived from the following four-potential:

$$\tilde{A}_{\mu}\,dx^{\mu} = -\frac{e}{2}\,\theta(V)\left(\frac{\rho^2 - \xi + w^{-2}}{S} - 1\right)\left(\frac{dV}{V} - \frac{dU}{U}\right) - $$
$$- 2e\,\theta(V)\,\frac{\rho\,d\rho}{\rho^2 + w^{-2}}. \qquad (8.2.27)$$

One can easily show that formulas (8.2.24) coincide with the expression for the field of a uniformly accelerated electric charge [see, e.g. Boulware (1980)]; the term proportional to $\delta(V)$ in (8.2.24) then reproduces correctly the term with a singularity at $V = 0$ introduced by Bondi and Gold (1955).

8.3. The Shift in the Self-Energy of a Charged Particle in the Field of a Black Hole

In this section we deal with the effect of changing of self-energy of a charged particle placed in a strong gravitational field. The effect is as follows. The total mass of a charged particle is composed of its 'mechanical' mass localized at the point where the charge is located, and the 'electromagnetic' mass spread over the region where electromagnetic field is nonzero. When a charged particle is placed in a nonuniform gravitational field, this field affects the 'local' and 'distributed' masses in a different manner, causing a 'deformation' of the electric field of the charge; the self-energy is additionally altered. The shift in self-energy depends on the position occupied by the particle. That is why forces acting in a gravitational field on particles with identical total inertial mass differ when one particles is charged while another is neutral.

The effect of the gravitational field on the self-energy of an electric charge was first analyzed by Fermi (1921). Fermi considered the case of an electric charge at rest in a uniform gravitational field and showed that the electromagnetic interaction causing a change in the inertial mass of a particle changes its gravitational mass by the same amount, in complete agreement with the equivalence principle. Arguments based on the equivalence principle do not work for the system as a whole in a nonuniform gravitational field. In general, one should expect the relationship between the self-energy of a charged particle and the change in the gravitational mass to be more complicated.

This is indeed the case for a particle in the field of a black hole. The necessary corrections (in the approximation $GM/c^2 \ll 1$) were found by Vilenkin (1979a). It was also shown in this paper that the inhomogeneity of the field leads to an additional force that pulls the charge away from the black hole. Still earlier, Unruh (1976a) proved that a similar force acts on a test charge placed in a thin hollow massive shell.

Smith and Will (1980) and Frolov and Zel'nikov (1980) noticed that the shift in the self-energy of an electric charge in the field of a Schwarzschild black hole can be calculated exactly, and carried out the calculation of the additional repulsive force. This result was later generalized to the case of Reissner–Nordström black holes

[Zel'nikov and Frolov (1982)], Kerr black holes [Leaute and Linet (1982)], and Kerr–Newman black holes [Lohiya (1982)].

To calculate the shift in the self-energy of a charge in the field of a black hole we assume that a classical particle, that is, a system of bound electric charges, is at rest on the symmetry axis in the stationary gravitational field; the equilibrium is maintained by an appropriate external force. Denote by $\xi^{\mu}_{(t)}$ a Killing vector field that is time-like at infinity where it is normalized by the condition $\xi^{\mu}_{(t)}\xi_{(t)\mu} = -1$. The energy of this system is[7]

$$E = -\int_{\Sigma} T_{\mu\nu}\xi^{\mu}_{(t)}\, d\sigma^{\nu}, \tag{8.3.1}$$

where $T_{\mu\nu}$ is the total metric energy-momentum tensor of the system. To specify the analysis, we choose a model of the charged particle: a stiff nonconducting thin sphere of mass m_0 and radius ε, with a charge e distributed over its surface. In view of the final transition to the limit of point-like particle, we assume ε to be much less than characteristic size of inhomogeneities of the gravitational and external electromagnetic fields; the terms $O(\varepsilon)$ in the final answer will be neglected.

The total energy E of a particle is made up of (1) E_0, that is, that part of energy which is connected with the 'mechanical' mass m_0 of the particle ($E_0 = |\xi_{(t)}\cdot\xi_{(t)}|^{1/2}m_0c^2$); (2) E_{self}, that is, the self-energy, or the energy of the self-action of the particle's charge; (3) E_{ext}, that is, the energy of interaction between the particle and the external field; (4) E_{int}, that is, the energy of the additional interaction that stabilizes the charged particle. The introduction of the additional interaction serves to satisfy the Laue theorem. If the equilibrium radius of a noncharged particle is denoted by ε_0, then $E_{\text{int}}(\varepsilon) = E_{\text{int}}(\varepsilon_0) + \frac{1}{2}K(\varepsilon - \varepsilon)^2$ and the changes (due to the introduction of the charge into the field) in the equilibrium size, $\Delta\varepsilon = \varepsilon - \varepsilon_0$, and in energy, $\Delta E = E_{\text{int}}(\varepsilon) - E_{\text{int}}(\varepsilon_0)$, can be made arbitrarily small by choosing a sufficiently high effective stiffness K. In what follows, the increments $\Delta\varepsilon$ and ΔE are neglected, that is, stiffness K is assumed to be sufficiently high. Having included the constant quantity $E_{\text{int}}(\varepsilon_0)$ into E_0, we can write the expression for the total energy of a particle as an expansion in powers of charge e of the particle:

$$E = E_0 + E_{\text{ext}} + E_{\text{self}}\,. \tag{8.3.2}$$

In the case of no external field and far from gravitating bodies, formula (8.3.2) reduces to the expression

$$E = mc^2, \tag{8.3.3}$$

where $m = m_0 + e^2/2\varepsilon c^2$. The difference between m and m_0 arises because of the field created by the charge. If a noncharged particle is introduced into a static gravitational field, the work done in this operation reduces the particle's energy to $E_0 = |\xi_{(t)}\cdot\xi_{(t)}|^{1/2}mc^2$. If the particle is charged, then part of the necessary work goes into changing the field around the particle. As a result, in general the sum $E_0 + E_{\text{self}}$ does not equal $|\xi_{(t)}\cdot\xi_{(t)}|^{1/2}mc^2$.

The expression for E_{self} can be recast to

$$E_{\text{self}} = \int_{\Sigma} \Pi^{\beta} d\sigma_{\beta}, \tag{8.3.4}$$

where

$$-\Pi^{\beta} = T^{\beta}_{\alpha}\xi^{\alpha}_{(t)} \equiv \frac{1}{4\pi}\left(F^{\beta}{}_{\mu}F_{\alpha}{}^{\mu} - \frac{1}{4}\delta^{\beta}_{\alpha}F_{\mu\nu}F^{\mu\nu}\right)\xi^{\alpha}_{(t)} \tag{8.3.5}$$

and $F_{\alpha\beta}$ is the strength of the field produced by the current

$$j^{\mu}(x) = \delta^{\mu}_{0}\frac{e}{4\pi\varepsilon^{2}\sqrt{-g(x)}}\,\delta(l(\mathbf{x}, \mathbf{x}_0) - \varepsilon). \tag{8.3.6}$$

Expression (8.3.6) describes the charge distribution of the particle. Here $l(\mathbf{x}, \mathbf{x}_0)$ is the invariant distance between a point $(t, \mathbf{x})$ and the center $(t, \mathbf{x}_0)$ of the charged particle along the geodesic connecting these points, and ε is the invariant size of the particle. The integration of (8.3.4) is carried out over a spacelike surface Σ that intersects the horizon H^+. Note that the integral (8.3.4) over the part of Σ within the event horizon is the energy of the field inside the black hole. Its contribution has already been taken into account in the definition of the total mass of the black hole. Therefore, when we are interested in calculating the energy shift in the field of a black hole with known parameters, we assume these parameters to be fixed and, hence, integrate (8.3.4) over that part of Σ which lies outside the black hole.[8]

Using Maxwell's equations

$$F^{\mu\nu}{}_{;\nu} = 4\pi j^{\mu}, \tag{8.3.7}$$

we can transform expression (8.3.5) to the form

$$\Pi^{\beta} = \frac{1}{8\pi}(\xi^{\nu}A_{\alpha}F^{\alpha\beta} - \xi^{\beta}A_{\alpha}F^{\alpha\nu} - 2\xi^{\alpha}A_{\alpha}F^{\nu\beta})_{;\nu} + \frac{1}{2}\xi^{\beta}A^{\alpha}j_{\alpha} - j^{\beta}\xi^{\alpha}A_{\alpha}. \tag{8.3.8}$$

Using Stoke's theorem, we can reduce the integral of the expression in parentheses in the right-hand side of (8.3.8) to the sum of the integral over the black hole surface and the integral over the infinitely removed one. The second of these integrals vanishes because the field falls off rapidly enough at infinity. It is readily shown that for a particle on the symmetry axis, the first integral (over the black hole surface) also vanishes. Finally, taking into account that ξ^{α} and j^{α} are parallel, we find

$$E_{\text{self}} = -\frac{1}{2}\int_{\Sigma} A_{\alpha} j^{\alpha}\xi^{\beta} d\sigma_{\beta}. \tag{8.3.9}$$

To obtain an explicit analytical expression for E_{self}, we can make use of the expression for the potential A_{α} produced by a point-like charge on the symmetry axis in the Kerr spacetime [it was found by Linet (1977a)]. This gives the following expression for E_{self}:

$$E_{\text{self}}^2 = \frac{e^2}{2\varepsilon}|g_{tt}|^{1/2} + \frac{e^2}{2}\frac{M}{r_0^2 + a^2}. \tag{8.3.10}$$

If the force necessary to fix this particle at the point r_0 is compared with that required to fix a neutral particle of mass $m = m_0 + e^2/2\varepsilon$, their difference proves to be

$$\Delta f = |\Delta f^\mu \Delta f_\mu|^{1/2} = e^2 \frac{M r_0}{(r_0^2 + a^2)^2}. \tag{8.3.11}$$

This excess force, Δf^μ, applied to the charged particle points along the symmetry axis away from the black hole.

If a charged particle (with a scalar, electric, or gravitational charge) is at rest in the vicinity of a black hole outside its symmetry axis, it is subject to an additional force [Gal'tsov (1982)]. This force is proportional to the angular momentum of the black hole and to the squared charge of the particle. It arises as a reaction to the tidal force exerted by the particle on the black hole and tending to slow down its rotation. This force vanishes if the particle rotates at the same angular velocity that the black hole has or if it is on the symmetry axis.

8.4. Mutual Transformation of Electromagnetic and Gravitational Waves in the Field of a Charged Black Hole

The effect of mutual transformation of electromagnetic and gravitational waves in an external electric field is a well-known corollary of the nonlinearity of Einstein–Maxwell equations [for a detailed discussion of this effect and the relevant bibliography, see, e.g., Sibgatullin (1984)].

This section will briefly describe this effect for the particular case of the propagation of photons and gravitons in the field of a charged black hole [Sibgatullin (1973, 1974, 1984), Sibgatullin and Alekseev (1974), Gerlach (1974, 1975)].

Let there be a metric $g_{\mu\nu}$ and an electromagnetic field A_μ that satisfy the system of Einstein–Maxwell equations, and let $h_{\mu\nu} = \delta g_{\mu\nu}$ and $a_\mu = \delta A_\mu$ be small perturbations against this background. Then the condition that $g_{\mu\nu} + h_{\mu\nu}$ and $A_\mu + a_\mu$ are also solutions of these equations yields the following linearized system of equations for perturbations:

$$k_{\mu\nu;\lambda}{}^{;\lambda} - k^\lambda{}_{\mu;\nu;\lambda} - k^\lambda{}_{\nu;\mu;\lambda} - \frac{1}{2} g_{\mu\nu} k_{;\lambda}{}^{;\lambda} -$$

$$- 2k^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} - \frac{1}{2} k_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + g_{\mu\nu} k^{\alpha\gamma} F_{\alpha\beta} F_\gamma{}^\beta +$$

$$+ kT_{\mu\nu} + 2F_\mu{}^\alpha f_{\nu\alpha} + 2F_\nu{}^\alpha f_{\mu\alpha} - g_{\mu\nu} F^{\alpha\beta} f_{\alpha\beta} = 0. \tag{8.4.1a}$$

$$f^{\alpha\beta}{}_{;\beta}-k_{\beta\rho}F^{\alpha\rho;\beta}-k_{\mu}{}^{\alpha}{}_{;\beta}F^{\mu\beta}-k_{\mu\beta}{}^{;\beta}F^{\alpha\mu}+\frac{1}{2}k_{,\beta}F^{\alpha\beta}=0. \tag{8.4.1b}$$

$$k_{\mu\nu}=h_{\mu\nu}-\frac{1}{2}hg_{\mu\nu},\quad h=h_{\alpha}{}^{\alpha},\quad k=k_{\alpha}{}^{\alpha},$$

$$f_{\mu\nu}=a_{\nu,\mu}-a_{\mu,\nu},\quad F_{\mu\nu}=A_{\nu,\mu}-A_{\mu,\nu}; \tag{8.4.2}$$

indices are lowered and raised and covariant differentiation is carried out using the metric $g_{\mu\nu}$;$T_{\mu\nu}$ is the energy-momentum tensor of the field $F_{\mu\nu}$ [see (A.48)].

This system is invariant with respect to the gauge transformations

$$h_{\mu\nu}\rightarrow h_{\mu\nu}-\xi_{\mu;\nu}-\xi_{\nu;\mu},$$

$$a_{\mu}\rightarrow a_{\mu}+\lambda_{;\mu}-\xi^{\alpha}A_{\mu;\alpha}-\xi^{\alpha}{}_{;\mu}A_{\alpha}. \tag{8.4.3}$$

The following additional conditions conveniently eliminate the arbitrariness of the gauge transformations:

$$k^{\alpha\beta}{}_{;\beta}=0,\quad a^{\alpha}{}_{;\alpha}=0. \tag{8.4.4}$$

We will consider the effect of the interaction between electromagnetic and gravitational perturbations due to those terms of (8.4.1a) that contain the electric field f and those terms of (8.4.1b) that contains the gravitational perturbation k. We use the approximation in which the wavelength λ of electromagnetic and gravitational waves is much shorter than the characteristic size L of inhomogeneities in the background fields $g_{\mu\nu}$ and A_{μ}. Using the geometrical optics approximation,[9] we can write the perturbations $k_{\mu\nu}$ and a_{μ} in the form

$$a_{\mu}=\mathrm{Re}[(\alpha_{\mu}+\varepsilon\beta_{\mu}+\cdots)\,e^{iS/\varepsilon}], \tag{8.4.5a}$$

$$k_{\mu\nu}=\mathrm{Re}[(\kappa_{\mu\nu}+\varepsilon\pi_{\mu\nu}+\cdots)\,e^{iS/\varepsilon}], \tag{8.4.5b}$$

where ε is a parameter characterizing the degree of smallness of a term with respect to a dimensionless parameter λ/L. The phase functions S in (8.4.5) are chosen to be identical. This can be done by redefining the preexponential factors if phases differ by a factor of order ε. Otherwise, if the difference between the phases S_a and S_k in the expressions for a_{μ} and k_{μ} is not small [$S_a-S_k=O(\varepsilon^0)$], the terms responsible for mixing enter with a high-frequency factor $\exp[S_a-S_k)/\varepsilon]$ and there is no mixing in the lowest order in ε.

If we denote $l_{\alpha}=S_{,\alpha}$ substitute (8.4.5) into equations (8.4.1) and into gauge conditions (8.4.4), and then set the terms of order ε^{-2} and ε^{-1} equal to zero, we arrive at the following relations: the eikonal equation

$$l_{\alpha}l^{\alpha}=0 \tag{8.4.6}$$

and the equations

$$l^{\mu}\kappa_{\mu\nu}=0,\quad l^{\mu}\alpha_{\mu}=0, \tag{8.4.7}$$

$$l^{\beta}{}_{;\beta}\alpha^{\mu}+2l^{\beta}\alpha^{\mu}{}_{;\beta}=N_{1}{}^{\mu}, \tag{8.4.8a}$$

$$l^{\beta}{}_{;\beta}\kappa_{\mu\nu}+2l^{\beta}\kappa_{\mu\nu;\beta}=N_{2\mu\nu}, \tag{8.4.8b}$$

$$N_{1}{}^{\mu}=l_{\beta}\left(\frac{1}{2}F^{\mu\beta}\kappa^{\alpha}{}_{\alpha}-F_{\gamma}{}^{\mu}\kappa^{\gamma\alpha}\right), \tag{8.4.9a}$$

$$N_{2\mu\nu}=-4\left[F_{\mu}{}^{\alpha}l_{[\nu}\alpha_{\alpha]}+F_{\nu}{}^{\alpha}l_{[\mu}\alpha_{\alpha]}+\frac{1}{2}g_{\mu\nu}F^{\alpha\beta}\alpha_{\alpha}l_{\beta}\right]. \tag{8.4.9b}$$

Condition (8.4.6) shows that the constant phase surface S = const is a null surface. The integral lines $x^{\mu}=x^{\mu}(\lambda)$ defined by the equation

$$\frac{dx^{\mu}}{d\lambda}=l^{\mu}(x) \tag{8.4.10}$$

and lying on this surface are, therefore, null geodesics and λ is an affine parameter.

Now complement the vector field l^{μ} to a complex null tetrad (l^{μ}, n^{μ}, m^{μ}, $\overline{m}^{\mu}$) and demand that the vectors of this tetrad be normalized by the conditions

$$l^{\mu}n_{\nu}=-1,\quad m^{\mu}\overline{m}_{\mu}=1 \tag{8.4.11}$$

(the remaining scalar products vanish) while the tetrads themselves be covariantly constant along the integral curves l^{μ}:

$$l^{\mu}m^{\nu}{}_{;\mu}=l^{\mu}n^{\nu}{}_{;\mu}=0. \tag{8.4.12}$$

The orthogonality condition $l^{\mu}m_{\mu}=0$ implies that the vectors m_{μ} and $\overline{m}_{\mu}$ are tangent to the surface S = const. It can be shown that the arbitrariness remaining in gauge transformations (8.4.3) [they preserve additional conditions (8.4.7)] can be used to rephrase the expressions for α_{μ} and $k_{\mu\nu}$ to the form

$$\alpha_{\mu}=A\overline{m}_{\mu}+\overline{A}m_{\mu},$$

$$\frac{1}{2}\kappa_{\mu\nu}=H\overline{m}_{\mu}\overline{m}_{\nu}+\overline{H}m_{\mu}m_{\nu} \tag{8.4.13}$$

[see, e.g., Sibgatullin (1984)]. Multiplying (8.4.8a) by m^{μ} and (8.4.8b) by $m^{\mu}m^{\nu}$ and denoting $\Phi_0=F^{\alpha\beta}l_{\alpha}m_{\beta}$ and $\rho=-\frac{1}{2}l^{\beta}{}_{;\beta}$, we arrive at the following system of equations:

$$\frac{dA}{d\lambda} - \rho A = \bar{\Phi}_0 H, \tag{8.4.14a}$$

$$\frac{dH}{d\lambda} - \rho H = -\Phi_0 A. \tag{8.4.14b}$$

These equations imply the relation

$$[l^\mu(|A|^2 + |H|^2)]_{;\mu} = 0 \tag{8.4.15}$$

that can be interpreted as the law of conservation of the total number of photons and gravitons.

The system of Equations (8.4.14) can be slightly simplified by transforming from the field variables A and H to the variables

$$\tilde{A} = zA, \quad \tilde{H} = zH, \tag{8.4.16a}$$

where

$$z = z(\lambda) \equiv z_0 \exp\left[-\int_{\lambda_0}^{\lambda} \rho \, d\lambda\right]. \tag{8.4.16b}$$

Equations rewritten in terms of these variables take the form

$$\frac{d\tilde{A}}{d\lambda} = \bar{\Phi}_0 \tilde{H}, \quad \frac{d\tilde{H}}{d\lambda} = -\Phi_0 \tilde{A}. \tag{8.4.17}$$

If $\Phi_0 = \bar{\Phi}_0$, this system reduces to a second-order equation

$$\frac{d^2\tilde{A}}{dx^2} + \tilde{A} = 0, \tag{8.4.18}$$

where $x = \int \Phi_0 \, d\lambda$. This equation shows that the amplitude of both electromagnetic and gravitational fields goes through oscillations due to the mutual transformations of photons and gravitons. The period of these oscillations, $\Delta\lambda$, is found from the condition [Sibgatullin (1974)]

$$2\pi = \int_{\lambda}^{\lambda+\Delta\lambda} \Phi_0 \, d\lambda. \tag{8.4.19}$$

The arguments given above are directly extended to the case of high-frequency photons and gravitons propagating in the field of a charged black hole. Eikonal equation (8.4.6),

$$g^{\mu\nu} S_{,\mu} S_{,\nu} = 0, \tag{8.4.20}$$

in the Riessner–Nordström metric (8.2.1) admits a complete integral

$$S = t \pm R(r) \pm \Psi(\theta) + m\phi, \tag{8.4.21}$$

where

$$R(r) = \int F^{-1}\sqrt{1 - b^2F/r^2}\ \mathrm{d}r,$$

$$\Psi(\theta) = \int\sqrt{b^2 - m^2/\sin^2\theta}\ \mathrm{d}\theta, \quad F = 1 - 2M/r + Q^2/r^2. \tag{8.4.22}$$

Light rays forming the surface S = const are parametrized by arbitrary constants b and m meaning the impact parameter and angular momentum, and are described by the equations

$$S = \text{const}, \quad \frac{\partial S}{\partial b} = \text{const}, \quad \frac{\partial S}{\partial m} = \text{const}. \tag{8.4.23}$$

For a given congruence of light rays, the affine parameter λ is related to r by the formula $\mathrm{d}\lambda = \mathrm{d}r(1-Fb^2r^{-2})^{-1/2}$; the complex null tetrad can be chosen to make Φ_0 a real quantity having the form

$$\Phi_0 = bQ/r^3. \tag{8.4.24}$$

Equation (8.4.19) for the oscillation period now takes the form

$$2\pi = Qb\int\frac{\mathrm{d}r}{r^3\sqrt{1 - Fb^2r^{-2}}}. \tag{8.4.25}$$

If high-frequency electromagnetic wave with amplitude A_{in} and impact parameter b is incident on a charged black hole, the result of its passing close to the black hole (unless the wave is captured) is the creation of outgoing electromagnetic and gravitational waves with amplitudes A_{out} and H_{out} [Sibgatullin (1974, 1984)]:

$$A_{\text{out}} = A_{\text{in}}\cos\left[2Qb\int_{r_0}^{\infty}\frac{\mathrm{d}r}{r^3\sqrt{1 - Fb^2r^{-2}}}\right],$$

$$H_{\text{out}} = A_{\text{in}}\sin\left[2Qb\int_{r_0}^{\infty}\frac{\mathrm{d}r}{r^3\sqrt{1 - Fb^2r^{-2}}}\right], \tag{8.4.26}$$

where r_0 is the minimal value of r for a light ray with a given impact parameter b. This value coincides with the maximal root of the equation

$$F(r) = r^2/b^2. \tag{8.4.27}$$

For $b = b_{\text{cr}}$, where

$$b_{\text{cr}}^2 = 4M^2[x + \frac{5}{2} + \sqrt{1 + 8x} + (8x)^{-1}(\sqrt{1 + 8x} - 1)].$$

$$x = 1 - Q^2/M^2, \tag{8.4.28}$$

this root equals

$$r_0 = r_{cr} = \frac{M}{2}(3 + \sqrt{1 + 8x}\,) \tag{8.4.29}$$

and becomes multiple. The integrals in (8.4.26) then diverge. The corresponding impact parameter corresponds to an unstable closed circular orbit.

Integrals in (8.4.26) diverge because the conditions of validity of the geometrical optics approximation are violated. The finite answer is obtained when the wave properties of light and gravitational radiation are taken into account. It is found that when $|b - b_{cr}| \ll 0(\omega r^{-1})$, the number of acts of mutual wave transformation close to an extremal ($Q = M$) black hole is of order unity and the total intensity of the outgoing electromagnetic and gravitational waves comes up to a considerable fraction of the intensity of the incoming electromagnetic radiation. The remaining energy is absorbed by the black hole.

If the black hole is rotating, the above-described effect of mutual transformation of photons and gravitons is accompanied by additional rotation of their planes of polarization [Sibgatullin (1984)]. Note that the effect of mutual transformation can be realized only in the neighborhood of a charged black hole. Although we have already mentioned that in realistic astrophysical conditions a black hole is unlikely to have a considerable charge, certain processes are nevertheless possible that produce a nonzero electric charge on a black hole. One of such processes caused by an external magnetic field acting on a black hole was described in the preceding chapter. Another possible process suggested by Shwarzman (1971) stems for the difference in the effect of radiative pressure on the electrons and ions of the matter accreting on a black hole.

8.5. Black Hole in an External Field. Interaction Between Black Holes

A black hole subjected to an external factor behaves to a certain extent as a compact elastic body. Some specifics of the 'response' of a black hole to this factor are mostly caused by the unambiguous dependence of the black hole size on its mass while the gravitational self-action of this mass is extremely high.

Before passing on to a detailed description of black-hole behavior in an external field, let us dwell on an aspect that often leads to misunderstanding. Imagine a situation of a lone (e.g., Schwarzschild) black hole at rest and a distant observer who at some moment of time switches on an external field in order to find the effect on the black hole. For instance, it can be a plane light wave directed at the hole. In general, the pressure of such a wave on an ordinary body results in the motion of the body (via the effects of light absorption and scattering). On the other hand, if we monitor the propagation of the light wavefront in the Schwarzschild metric, we discover that infinitely long time (by the clock of the distant observer) is necessary

for the wavefront to reach the gravitational radius. A detailed calculation shows [e.g., see Hanni (1977)] that the wavefront bends around the black hole after which the wave propagates further. After some time, a divergent component appears, indicating scattering, and a new wavefront forms around the black hole; it corresponds to the radiation falling into it. The question is: How can a black hole 'feel' the pressure of radiation and start moving in a finite time (by the clock of distant observer) if for this observer the radiation never reaches the event horizon?[10] To answer this question, it is instructive to analyze a similar situation that arises when a light wave is scattered by a body of radius r_0, composed of matter whose refractive index n increases continuously inward from 1 at the boundary (at $r = r_0$) to infinity on some surface inside this body (at $r = r_1$). As in the case of a black hole, the process of light propagation to the surface $r = r_1$ may take infinitely long time. However, the body starts moving before this process is completed. It can be shown that once the wave front has reached the surface $r = r_0$, the energy-momentum flux across this surface becomes nonzero in the general case, and the force due to this flux causes the motion of the body as a whole.

Likewise, if a black hole is inclosed in an imaginary sphere of radius $r_0 \gtrsim r_g$, a light wave produces a nonzero energy-momentum flux across this surface and the entire region within r_0 (the black hole and 'part of the space' surrounding it) starts moving with respect to the external observer. The energy of an ordinary light wave in a volume of order r_g^3 is much smaller than $r_g c^4/G$, so that this wave produces a negligible effect on the black hole metric. As a result, in the immediate vicinity of the black hole all phenomena unfold as if there were no radiation (in the reference frame fixed to the black hole). On the contrary, a distant observer will be able to detect the motion of this reference frame with respect to the observer's.

Although the general problem of the motion of a black hole in an external field does not admit an analytical solution, a detailed description of the motion is possible in terms of a sort of perturbation theory in the particular case of a black hole not interacting too strongly with its surroundings. This description can be given, for example, for the motion in an external gravitational field if the characteristic size of the black hole, determined by its mass M, is much less than the characteristic scale L of inhomogeneities of the gravitational field in which the black hole moves.[11] In this case the external gravitational field slightly changes the metric in the neighborhood of the black hole. Hence, the metric $g_{\mu\nu}$ in the region outside the black hole, determined by the characteristic scale M, can be written in the form

$$g_{\mu\nu} = \overset{[0]}{g}_{\mu\nu} + \varepsilon \overset{[1]}{g}_{\mu\nu} + \varepsilon^2 \overset{[2]}{g}_{\mu\nu} + \cdots, \tag{8.5.1}$$

where $\overset{[0]}{g}_{\mu\nu}$ is the metric of the unperturbed black hole (the Kerr metric) and $\varepsilon = M/L$. Likewise, the effect of the black hole on the external metric on the scale of order L can be considered small; it can be compensated for by corrections, with the external metric written in the form

$$g_{\mu\nu} = \overset{(0)}{g}_{\mu\nu} + \varepsilon \overset{(1)}{g}_{\mu\nu} + \varepsilon^2 \overset{(2)}{g}_{\mu\nu} + \cdots \tag{8.5.2}$$

The assumption of the absence in the black hole neighborhood of matter whose fall can result in a sufficiently fast modification of its parameters, and the conditions of weakness of the black hole-external field interaction imply that expansions (8.5.1) and (8.5.2) have a common domain of applicability. In other words, both expansions hold simultaneously in an interval of distances from the black hole that is determined by a characteristic scale $l \sim \varepsilon^{\alpha} M$ $(0 < \alpha < 1,\ M \ll l \ll L)$. The expansions themselves are found unambiguously by comparing (matching) them in this common region.

The method of matching the asymptotic expansions for analyzing the motion of black holes in external fields and the interaction between the black holes was developed by D'Eath (1975a, b, 1978, 1979). The following results were obtained by this method in the problem of interest here [Demyansky and Grishchunk (1974), D'Eath (1975a,b, 1979), Kates (1980, 1981), Damour (1983), Thorne and Hartle (1985)].

The term $g^{[1]}_{\mu\nu}$ in expansion (8.5.1) vanishes. The mass M, momentum P, and angular momentum J of a black hole can be determined from the parameters of the unperturbed metric $g^{[0]}_{\mu\nu}$ even though the spacetime as a whole need not be asymptotically flat. By an order of magnitude, the uncertainties in these parameters due to the terms $\varepsilon^2 g^{[2]}_{\mu\nu}$ are

$$\Delta M \sim \varepsilon^2 M, \quad \Delta P \sim \varepsilon^2 M, \quad \Delta J \sim \varepsilon^2 M^2. \tag{8.5.3}$$

The external space metric $g^{(0)}_{\mu\nu}$ is everywhere regular but the corrections to it, formally extended to the entire space, reveal singularities, corresponding to the motion of the black hole, on the time-like world lines γ. Coordinates t, x^i can be introduced in the neighborhood of γ, such that the metric takes the form

$$\begin{aligned} \overset{(0)}{g}_{00} &= -1 - E_{ij}x^i x^j + \cdots, \\ \overset{(0)}{g}_{0i} &= \frac{2}{3}\varepsilon_{ijk}B^j{}_l x^k x^l + \cdots, \\ \overset{(0)}{g}_{ij} &= \delta_{ij}(1 - E_{kl}x^k x^l) + \cdots \end{aligned} \tag{8.5.4}$$

(here $E_{ij} = R_{i0j0}$ and $B_{ij} = \frac{1}{2}\varepsilon_{ikl}R^{kl}{}_{j0}$ are the 'electric' and 'magnetic' components of the curvature tensor).

A change at the part of perturbation $g_{\mu\nu} - g^{(0)}_{\mu\nu}$ which is regular on the trajectory of the black hole leads to a small uncertainty in the determination of E_{ij} and B_{ij}.

The trajectory of the black hole in the lowest approximation in ε is a geodesic, with the spin of the black hole transported parallel to itself along this line [D'Eath (1975a)]. The corrections describing the deviations in a motion of a black hole from a geodesic and from the Fermi–Walker transport of spin (A.12) [see Thorne and Hartle (1985)] are given by the following equations

$$\frac{dP^i}{dt} = -B^i_j J^j, \tag{8.5.5}$$

$$\frac{dJ^i}{dt} = -\varepsilon^i_{jk} E^k_l \frac{1}{M} J^j J^l . \tag{8.5.6}$$

The changes in the momentum and the spin of the black hole that follow from these equations greatly exceed the uncertainties ΔP^i and ΔJ^i of (8.5.3),

$$\begin{aligned} \Delta P^i &\sim \frac{1}{M} \varepsilon^i_{jk} B^j_l J^k J^l , \\ \Delta J^i &\sim M^2 E^i_j J^j , \end{aligned} \tag{8.5.7}$$

caused by changes in E^i_j and B^i_j. In principle, therefore, these effects of nongeodesic motion of a rotating black hole and of the precession of its angular momentum are observable.

It should be emphasized that Equations (8.5.5) and (8.5.6) coincide in form with the equations of motion of rotating test particles in an external gravitational field. It is important that when the extremely strong gravitational self-action of black holes is taken into account, the form of equations is not changed. To a distant observer, a black hole continues to move in the external field in the same manner as a small test particle did.

Likewise, it can be shown [Bicak (1980)] that when the external metric field E acts on a black hole of charge Q and mass M, the hole acquires an acceleration $a = QE/M$.[12] It is interesting to note that Ernst (1976b) obtained an exact solution of the Einstein–Maxwell equations describing the motion of a charged black hole in a uniform electric field. The corresponding metric in retarded coordinates is

$$\begin{aligned} ds^2 &= B^2(-H\, du^2 - 2du\, dr - 2wr^2\, du\, dx + r^2 G^{-1}\, dx^2) + \\ &\quad + B^{-2} r^2 G\, dz^2 , \end{aligned} \tag{8.5.8}$$

where

$$\begin{aligned} B &= 1 + QE_0 x + \frac{1}{4} E_0^2 (r^2 G + Q^2 x^2), \\ G &= 1 - x^2 - 2\, Mwx^3 - Q^2 w^2 x^4, \\ H &= -w^2 r^2 G + wr \frac{dG}{dx} + 1 + 6\, Mwx + 6\, Q^2 w^2 x^2 - \\ &\quad - 2(M + 2\, Q^2 wx) r^{-1} + Q^2 r^{-2} . \end{aligned} \tag{8.5.9}$$

Here w is the acceleration of the black hole and E_0 is the strength of the external electric field. If the condition $QE_0 = Mw$ is satisfied, there are no nodal singularities that metric (8.5.8) has in the general case. Note that the metric describing the accelerated motion of a body is not explicitly time-dependent because of the

appropriate choice of coordinates. The flat space metric in the Rindler coordinates, connected with the motion of observers at constant acceleration, has a similar property.

The method of matching of asymptotic expansions (see above) also serves to analyze the interaction between two black holes. If the distance between holes greatly exceeds their graviational radii and the holes move with respect to each other at a velocity much lower than the speed of light, the corresponding equations of motion of such interacting black holes were obtained by D'Eath (1975b, 1979) [see also Thorne and Hartle (1985)]. The gravitational field in the neighborhood of each black hole is described by a perturbed Kerr metric. The metric far from black holes is found using the post-Newtonian approximation to a required order of accuracy. Matching these expansions yields the following system of equations of motion of one of the black holes and of the precession of its angular momentum in the field of the other black hole:

$$m_1 \frac{d^2 \mathbf{x}_1}{dt^2} = \mathbf{F}_1^{(1)} + \mathbf{F}_1^{(2)} + O(\varepsilon^4), \tag{8.5.10a}$$

$$\frac{d\mathbf{J}_1}{dt} = [(\Omega_1^{(1)} + \Omega_1^{(2)} + \Omega^{(3)}) \times \mathbf{J}_1]. \tag{8.5.10b}$$

Here and throughout this chapter we denote: x_i is the position and $\mathbf{v}_i$ is the velocity of the ith black hole having a mass M_i and an angular momentum $\mathbf{J}_i$; $\mathbf{r}_{21} = \mathbf{x}_2 - \mathbf{x}_1$, $\mathbf{v}_{21} = \mathbf{v}_2 - \mathbf{v}_1$ are the position and velocity of the second black hole with respect to the first;

$$r = |\mathbf{r}_{21}|, \quad \mathbf{n} = \mathbf{r}_{21}/r; \quad J_i = |\mathbf{J}_i| \text{ and } \mathbf{j}_i = \mathbf{J}_i/J_i$$

are the magnitude and the unit vector in the direction of angular momentum of the ith black hole. The smallness parameter is equal to the ratio of the maximal among the gravitational radii to the characteristic distance between black holes. The value of the force $\mathbf{F}_1^{(1)}$ corresponding to the geodesic law of motion of one body in the gravitational field created by the second body [it was found by Einstein, Infeld, and Hoffman (1938)] is

$$\mathbf{F}_1^{(1)} = \frac{M_1 M_2}{r^2} \left\{ \mathbf{n} \left(1 - \frac{4M_2 + 5M_1}{r} + \mathbf{v}_1^2 + 2\mathbf{v}_2^2 - 4\mathbf{v}_1\mathbf{v}_2 - \frac{3}{2}(\mathbf{v}_2\mathbf{n})^2 \right) - \right.$$

$$\left. - \mathbf{v}_{21}[\mathbf{n}(3\mathbf{v}_2 - 4\mathbf{v}_1)] \right\}. \tag{8.5.11}$$

The term $\mathbf{F}_1^{(2)}$ in (8.5.10a),

$$\mathbf{F}_1^{(2)} = \frac{M_1 J_2}{r^3} \{6\mathbf{n}([\mathbf{j}_2 \times \mathbf{n}]\mathbf{v}_{12}) + 4[\mathbf{j}_2 \times \mathbf{v}_{12}] - 6[\mathbf{j}_2 \times \mathbf{n}](\mathbf{v}_{12}\mathbf{n})\} +$$

$$+ \frac{M_2 J_1}{r^3} \{6\mathbf{n}([\mathbf{j}_1 \times \mathbf{n}]\mathbf{v}_{12}) + 3[\mathbf{j}_1 \times \mathbf{v}_{12}] - 3[\mathbf{j}_1 \times \mathbf{n}](\mathbf{v}_{12}\mathbf{n})\}, \qquad (8.5.12)$$

describes the additional force due to spin-orbital interaction. The term $O(\varepsilon^4)$ in the same equations corresponds to the spin-spin interaction and to the interaction of the quadrupole moment of the black hole with the curvature: Both are of order ε^4.

Equation (8.5.10b) describes the precession of the angular momentum of a black hole with respect to comoving orthogonal reference frame that does not rotate with respect to an infinitely distant observer. The gravimagnetic ($\Omega_1^{(1)}$) and the geodesic ($\Omega_1^{(2)}$) components of the angular velocity of this precession and the component $\Omega_1^{(3)}$ due to the coupling of the quadrupole moment of the black hole to curvature are, respectively,

$$\Omega_1^{(1)} = \frac{1}{r^3}[-\mathbf{J}_2 + 3\mathbf{n}(\mathbf{n}\mathbf{J}_2)],$$

$$\Omega_1^{(2)} = \frac{M_2}{r^2} \left[\left(\frac{3}{2}\mathbf{v}_1 - 2\mathbf{v}_2\right) \times \mathbf{n}\right], \qquad (8.5.13)$$

$$\Omega_1^{(3)} = \frac{3M_2}{M_1 r^3} \mathbf{n}(\mathbf{n}\mathbf{J}_1).$$

In the limiting case of $M_1 \ll M_2$, these formulas coincide with the equations of motion of a rotating test particle in the field of a massive rotating body. [A detailed description of the solution of this last problem can be found in the monograph of Misner, Thorne, and Wheeler (1973) which also gives references to a numerous original publications.]

For more details on the problem of motion of black holes in a given field and on interaction between them, see Thorne, Price, and MacDonald (1986).

In the opposite case of the relative velocity v of two black holes being close to the speed of light, the expansion in the small parameter γ^{-1} can be used [$\gamma = (1-v^2)^{-1/2}$]. This method was employed by D'Eath (1975b, 1979) to solve the problem of the scattering of two ultrarelativistic black holes moving on parallel oppositely directed courses. The calculation started with the expression for the metric of a lone nonrotating black hole moving at a constant velocity, in the limit $v \to c$. This metric can be obtained by transformations for the Schwarzschild metric written in isotropic coordinates:

$$ds^2 = -\left(\frac{1 - M/2r}{1 + M/2r}\right)^2 dt^2 + \left(1 + \frac{M}{2r}\right)^4 (dx^2 + dy^2 + dz^2), \tag{8.5.14}$$

where $r^2 = x^2 + y^2 + z^2$, and M is the black hole mass. Applying the Lorentz transformation

$$\bar{t} = (1 - v^2)^{-1/2}(t + vx), \quad \bar{y} = y,$$

$$\bar{x} = (1 - v^2)^{-1/2}(x + vt), \quad \bar{z} = z, \tag{8.5.15}$$

we rewrite (8.5.14) in the form

$$ds^2 = (1 + A)^2(-d\bar{t}^2 + d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2) +$$

$$+ \left[\left(\frac{1 - A}{1 + A}\right)^2 - (1 + A)^4\right]\frac{(d\bar{t} - v\, d\bar{x})^2}{(1 - v^2)}, \tag{8.5.16}$$

where

$$A = \frac{M}{2r} = \frac{\mu(1 - v^2)}{2[(\bar{x} - v\bar{t})^2 + (1 - v^2)(\bar{y}^2 + \bar{z}^2)]^{1/2}},$$

$$\mu = \gamma M = M/\sqrt{1 - v^2}. \tag{8.5.17}$$

Taking the limit $v \to 1$ in (8.5.16) at a constant value of μ and denoting $\bar{t} - \bar{x} = \bar{v}$, $\bar{t} + \bar{x} = \bar{u}$, we find [Aichelburg and Sexl (1971)]

$$ds^2 = -d\bar{v}\, d\bar{u} + d\bar{y}^2 + d\bar{z}^2 + 4\mu\left[\frac{1}{|\bar{v}|} - \delta(\bar{v}) \ln(\bar{y}^2 + \bar{z}^2)\right] d\bar{v}^2. \tag{8.5.18}$$

Using the coordinate transformation $du = d\bar{u} - 4\mu dv/|\bar{v}|v = \bar{v}$, the above expression is converted to the form

$$ds^2 = -du\, dv + d\bar{y}^2 + d\bar{z}^2 - 4\mu\, \delta(v) \ln(\bar{y}^2 + \bar{z}^2)\, dv^2. \tag{8.5.19}$$

We see from this form of the metric that the corresponding gravitational field is a special case of the axially symmetrical plane gravitational wave concentrated on the interface $v = 0$ between two flat half-spaces, $v > 0$ and $v < 0$. The limiting procedure carried out above changes the algebraic type of the Weyl tensor; metric (8.5.19) is of type N, instead of the type D characterizing the original metric (8.5.14).[13] The corresponding curvature tensor vanishes everywhere except the surface $v = 0$ where its nonzero components have singularities of type $\delta(v)$.

If there are two ultrarelativistic black holes moving toward each other on parallel courses, their gravitational field being concentrated in two plane waves described by metrics (8.5.19), the interaction between them disturbs these gravitational waves which are scattered in the transmission through each other. D'Eath (1978) was able to show that if the impact parameter is comparable with the quantity $M\gamma^2$, where M is the characteristic mass of black holes and γ is the typical Lorentz factor in the center-of-mass reference frame, narrow beams of gravitational radiation of solid angles $\sim\gamma^2$ and characteristic power ~1 (in units of c^5/G) appear along the directions of motion of the black holes. The radiation is caused by the rapidly changing acceleration of a black hole at the moment of passage at the shortest distance from its counterpart.

If the impact parameter is comparable with the quantity $\mu = M\gamma$, gravitational radiation along the direction of motion remains weak up to angles $\theta \lesssim \gamma^{-1}$. At larger angles (in the interval $\gamma^{-1} \ll \theta \ll 1$), the total energy of gravitational radiation per unit solid angle is about $\mathrm{d}E/\mathrm{d}\Omega \approx 0.248\ M\gamma/2\pi$. D'Eath (1978) assumed that the radiation of two black holes of equal mass in head-on collision is fairly isotropic and concluded that the efficiency of the transformation of the energy of black holes, 2μ, into the radiation energy, ΔE, is close to 25%: $\Delta E/2\mu \approx 0.248$ [Smarr *et al.* (1976); see also Smarr (1977, 1979) Bovyn (1983)].

The general restriction on the maximal efficiency of conversion of the energy of black holes in head-on collision into the gravitational radiation energy ΔE can be found on the basis of Hawking's theorem [Penrose (1974), Smarr *et al.* (1976)]. If the two black holes have identical mass M and their center-of-mass velocity towards each other at infinity equals v the maximal efficiency $\varepsilon = \Delta E/2\mu$ is

$$\varepsilon \le 1 - \sqrt{\frac{1 - v^2}{2}}.$$

The estimates given above show that the actual efficiency for ultrarelativistic black holes is ≈25% of the maximal value $\varepsilon(v = 1) = 1$. If nonrelativistic black holes collide, the efficiency reduces by nearly two orders of magnitude. The numerical evaluation of the energy of gravitational radiation in a head-on collision of two nonrotating black holes with identical mass and zero relative velocity at infinity gives [Smarr (1979); see also Petrich *et al.* (1985)]

$$\Delta E = 2.5\ 10^{-3} M.$$

Recall (see Section 3.3) that the amount of radiated energy in the case of radial fall of a *test* particle of mass m at the parabolic velocity into a black hole is given by the formula

$$\Delta E = 0.01\ m^2/M.$$

Note that the same formula reproduces the numerical result given above quite well if the reduced mass of two black holes, $M/2$, is substituted for m.

An external field causes a deformation of a black hole. We will briefly outline the changes induced in the properties of a black hole when it is 'inserted' into a

gravitational field produced by a distribution of matter. If the black field is not rotating and the gravitational field is axially symmetric, this problem admits a fairly complete solution [Israel and Khan (1964), Doroshkevich *et al.* (1965), Mysak and Szekeres (1966), Israel (1973), Geroch and Hartle (1982)]. A generalization to the case of a rotating black hole can be found in Tomimatsu (1984).

Recall first of all that a static axially symmetric vacuum gravitational field is described by the Weyl metric

$$ds^2 = -e^{2U} dt^2 + e^{-2U}[e^{2V}(d\rho^2 + dz^2) + \rho^2 d\phi^2], \tag{8.5.20}$$

where U and V are functions of ρ and z and satisfy the equations

$$U_{,\rho\rho} + \frac{1}{\rho} U_{,\rho} + U_{,zz} = 0, \tag{8.5.21a}$$

$$V_{,\rho} = \rho(U^2_{,\rho} - U^2_{,z}), \quad V_{,z} = 2\rho U_{,\rho} U_{,z}. \tag{8.5.21b}$$

[see, e.g., Kramer *et al.* (1980)]. It is rapidly verified that system (8.5.21b) is integrable if condition (8.5.21a) is satisfied. The vacuum metric (8.5.20) is uniquely determined by choosing the solution of Equation (8.5.21a) because a solution of system (8.5.21b) for a given function $U(\rho,z)$ is easily found in quadrature. Thus the Schwarzschild metric in these coordinates corresponds to the following solution

$$U = U_S \equiv \frac{1}{2} \ln\left(\frac{\lambda - 1}{\lambda + 1}\right), \quad V = V_S \equiv \frac{1}{2} \ln\left(\frac{\lambda^2 - 1}{\lambda^2 - \mu^2}\right), \tag{8.5.22}$$

where

$$\lambda = \frac{R_+ + R_-}{2M}, \quad \mu = \frac{R_+ - R_-}{2M}, \quad R_\pm = \sqrt{\rho^2 + (z \pm M)^2}\,; \tag{8.5.23}$$

the event horizon H is now defined by the condition

$$\rho = 0, \ -M \le z \le M. \tag{8.5.24}$$

In view of the uniqueness theorem (see Section 6.3), this solution is unique for a static black hole in vacuum in an asymptotically flat spacetime. Any other static *vacuum* solution having a regular horizon [14] cannot be asymptotically flat.

Strictly speaking, therefore, a perturbed black hole is necessarily described by a nonvacuum solution of Einstein's equations. In the simplest case, we can assume that the matter creating the external gravitational field is located at some distance from the black hole. Then the gravitational field in the vicinity of its horizon is described by the vacuum Weyl metric (8.5.20). Exact solution of such complete problems can be found only in some very particular cases of matter distribution. Nevertheless, an analysis of the properties of vacuum Weyl solution close to a regular horizon yields detailed information on the possible effect of external factors on the black hole surface.[15]

The solution describing a perturbed black hole can be written in the following form [Geroch and Hartle (1982)]:

$$U = U_S + \hat{U}, \quad V = V_S + \hat{V}, \tag{8.5.25}$$

where U_S and V_S are given by (8.5.22) and $\hat{U}$ is the solution of the homogeneous equation (8.5.21a) that satisfies the regularity condition on the segment $\rho = 0$, $-M \le z \le M$ and in a certain neighborhood of this segment; $\hat{U}$ takes on identical values at the segment ends:

$$\hat{U}(\rho = 0,\ z = -M) = \hat{U}(\rho = 0,\ z = M) \equiv u. \tag{8.5.26}$$

The value of $\hat{V}$ is uniquely determined from Equation (8.5.21b) provided $V = 0$ on the parts of the axis $\rho = 0$ that lie outside the matter-occupied region. Condition (8.5.26) reflecting the absence of nodal singularities guarantees that the net force applied by the external field to the black hole as a whole vanishes; hence, a regular static solution is possible. Using the second equation in (8.5.21b), one can prove that the following relation holds at the event horizon:

$$\hat{V} = 2\hat{U} - 2u. \tag{8.5.27}$$

Using the coordinate transformation

$$\rho = e^u r \sqrt{1 - \frac{2M_0}{r}} \sin\theta, \quad z = e^u (r - 2M_0)\cos\theta, \tag{8.5.28}$$

$$M_0 = M\, e^{-u}, \tag{8.5.29}$$

it is possible to recast metric (8.5.20) to the form

$$ds^2 = -e^{-2\hat{U}} \left(1 - \frac{2M_0}{r}\right) dt^2 + e^{2\hat{V} - 2\hat{U} + 2u} \times$$

$$\times \left[\left(1 - \frac{2M_0}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + e^{-2\hat{V}} \sin^2\theta\, d\phi^2)\right]. \tag{8.5.30}$$

The event horizon H is described in these coordinates by the equation $r = 2M_0$ and the two-dimensional metric on its surface becomes, after (8.5.27) is taken into account,

$$ds^2 = 4 M_0^2 (e^{2\hat{U} - 2u} d\theta^2 + e^{-2\hat{U} + 2u} \sin^2\theta\, d\phi^2). \tag{8.5.31}$$

It is easily found that the horizon surface is a sphere deformed in an axially symmetric manner [16] whose area is

$$A = 16\pi M_0^2. \tag{8.5.32}$$

To a certain extent, the action of the external gravitational field, on a black hole, with $\exp\hat{U}$ playing the role of potential, is analogous to the effect produced on an elastic massive body. In order to verify this statement, consider the case of the values of $\hat{U}$ being higher at the poles that at the equator. In this situation any test body at rest near a pole is subject to a force that tends to move the body toward the equator. Correspondingly, the horizon surface is found to be formulated at the poles, as follows from (8.5.31). The surface gravity κ is constant at the horizon surface:

$$\kappa = e^{u}/4M_0. \tag{8.5.33}$$

The difference between the value of mass M observed at infinity and the value M_0 determining the area of the event horizon and playing the role of irreducible mass (see Section 8.1) is explained in the following manner. Consider the process of 'inserting' a Schwarzschild black hole (obviously, with $M = M_0$) into a given external gravitational field. If this process is carried out slowly enough, the area A of the black hole surface remains constant; hence, M_0 is preserved in this process. On the contrary, the quantity $M = e^{u}M_0$ undergoes a change when the gravitational potential e^{u} of the external field changes at the point where the black hole is located. The difference $M - M_0 = M_0(e^{u} - 1)$ is the work done by the external gravitational field when the black hole is 'inserted' into it. It can be shown [Geroch and Hartle (1982)] that if the strong energy condition is satisfied (see Appendix), the potential u assumes only nonpositive values.

As an example illustrating these general arguments, consider the explicit expression for the metric that describes a black hole in an external quadrupole field [Doroshkevich *et al.* (1965)]:

$$\hat{U} = \frac{1}{4}q(3\lambda^2 - 1)(3\mu^2 - 1),$$

$$\hat{V} = -3q\lambda(1-\mu^2) - \frac{9}{16}q^2(\lambda^2 - 1)(1-\mu^2)(9\mu^2\lambda^2 - \lambda^2 - \mu^2 + 1), \tag{8.5.34}$$

where q is a parameter characterizing the quadrupole moment of the system that produces the external gravitational field. At the horizon we have $\lambda = 1$, $u = z/M$, so that we immediately obtain that in the case under consideration $u = q$. The Gaussian curvature of the event horizon is

$$K = \frac{e^{q}}{4M_0^2}(1 + 3q - 12q\mu^2 - 9q^2\mu^2 + 9q^2\mu^4). \tag{8.5.35}$$

A constant external quadrupole field described by the solution (8.5.4) can be created by distant masses at rest. This solution also describes approximately the effect produced on a black hole by distant free masses whose velocities due to the mutual attraction are initially small and whose field in nearly static.

We have already indicated that the main difficulty in deriving exact complete axially symmetric stationary solutions, describing the behavior of a black hole in a

gravitational field, lies in finding the solution in the region with matter. The cases of a black hole in a uniform electric field [Ernst (1976b)] and in a uniform magnetic field [Ernst (1976a), Ernst and Wild (1976), Gal'tsov and Petukhov (1978), Gal'tsov (1980), Wild and Kerns (1980), Aliev *et al.* (1980), Wild *et al.* (1981), Krori *et al.* (1983, 1984), Dadlich (1983), Dhuraudhar and Dadlich (1984a,b)] are important exceptions.

An interesting possibility of studying the interaction between black holes is found in analyzing so-called *instantly static configurations* that describe a system of interacting black holes at the moment when all of them are at rest [Misner and Wheeler (1957), Misner (1960, 1963), Lindquist (1963), Brill and Lindquist (1963), Gibbons (1972), Bowen and York (1980), Kulkarni *et al.* (1983), Bowen *et al.* (1984), Kulkarni (1984)]. This possibility arises because the spacetime metric $g_{\mu\nu}$ at the moment of temporal symmetry can be so chosen that the conditions $g_{0i} = 0$, $\partial_0 g_{00} = \partial_0 g_{ij} = 0$ are satisfied and the three-dimensional geometry of space is found by solving the equation

$$^{(3)}R = 0, \tag{8.5.36}$$

where $^{(3)}R$ is the scalar of the three-dimensional curvature of metric h_{ij}. In the presence of electromagnetic field, only the components $E_i = F_{i0}$ are nonzero at the moment of temporal symmetry. The initial conditions at this moment of time for the Einstein–Maxwell system of equations are

$$^{(3)}R = 2E_i E^i, \quad E^i_{\ :i} = 0. \tag{8.5.37}$$

Equations (8.5.37) describing a system of N interacting charged black holes have the following solution [Lindquist (1963), Brill and Lindquist (1963)]:

$$\mathrm{d}l^2 \equiv h_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j = (\chi\psi)^2(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2),$$

$$E_i = [\ln(\chi/\psi)]_{,i}\,, \tag{8.5.38}$$

where

$$\chi = 1 + \sum_{i=1}^{N} \frac{\alpha_i}{|\mathbf{r} - \mathbf{r}_i|}, \qquad \psi = 1 + \sum_{i=1}^{N} \frac{\beta_i}{|\mathbf{r} - \mathbf{r}_i|},$$

$$\alpha_i > 0, \;\; \beta_i > 0, \; \mathbf{r} = (x, y, z).$$

(The two-dimensional section of the corresponding metric is schematically shown in Figure 73 for the case of $N = 3$.) The mass M_i and charge Q_i of ith black hole, determined by the asymptotic behavior of the solution at infinity on the sheet Σ_i (as $\mathbf{r} \to \mathbf{r}_i$) are

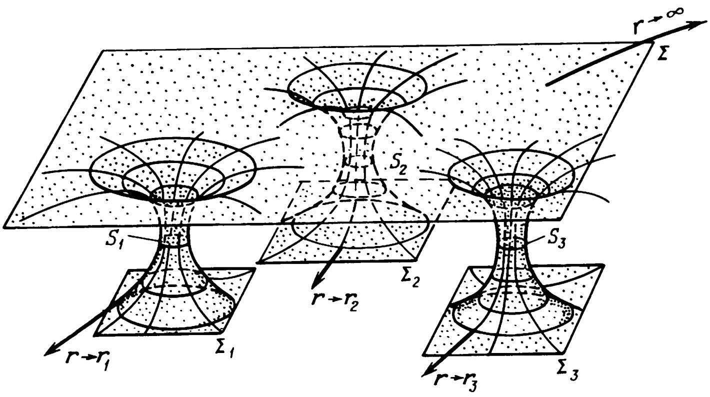


Fig. 73. Instantly static configuration of three interacting black holes (two-dimensional spatial section of spacetime).

$$M_i = \alpha_i + \beta_i + \sum_{\substack{i,j=1 \\ i\neq j}}^{N} \frac{\alpha_i\beta_j + \alpha_j\beta_i}{|\mathbf{r}_i - \mathbf{r}_j|}, \tag{8.5.39a}$$

$$Q_i = \beta_i - \alpha_i + \sum_{\substack{i,j=1 \\ i\neq j}}^{N} \frac{\beta_i\alpha_j - \alpha_i\beta_j}{|\mathbf{r}_i - \mathbf{r}_j|}. \tag{8.5.39b}$$

The mass M and charge Q of a system of interacting black holes, determined by the asymptotic behavior of the solution at infinity on the sheet Σ (as $r \to \infty$) are

$$M = \sum_{i=1}^{N}(\alpha_i + \beta_j), \quad Q = \sum_{i=1}^{N}(\beta_i - \alpha_i). \tag{8.5.40}$$

[If $\beta_i = \alpha_i$, solution (8.5.38) describes a system of noncharged black holes; see Misner and Wheeler (1957), Misner (1960, 1963), Gibbons (1972).]

It is easily shown that

$$Q = \sum_{i=1}^{N} Q_i, \quad M_{\text{int}} \equiv M - \sum_{i=1}^{N} M_i = -\sum_{\substack{i,j=1 \\ i\neq j}}^{N} \frac{\alpha_i\beta_j + \alpha_j\beta_i}{|\mathbf{r}_i - \mathbf{r}_j|} < 0. \tag{8.5.41}$$

These relation show that the charges of black holes sum up additively while the gravitational mass defect due to the interaction between black holes reduces the total mass of the system below the sum of the masses of component holes. For a discussion of the properties of two-dimensional surfaces corresponding to the position of apparent horizon at the moment of time symmetry for a system of interacting black holes, see, for example, Brill and Lindquist (1963), Gibbons (1972), Bishop (1984).

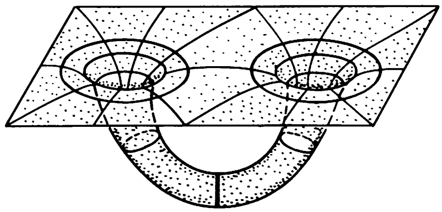


Fig. 74. Solution of 'wormhole' type (two-dimensional spatial section of spacetime).

Misner (1960, 1963) and Lindquist (1963) generalized solution (8.5.38) to the case of a system of black holes replaced by a set of 'wormholes' (a two-dimensional section of such a space is schematically shown in Figure 74 in the case of a single 'wormhole').

It should be emphasized again that in general a system of black holes described by solution (8.5.38) cannot be at rest at all times. The situation when $\alpha_i = 0$ and the condition $M_i = Q_i$ holds for all black holes is an exception [Hartle and Hawking (1972)]. If $\alpha_i \neq 0$, expression (8.5.38) can be used as the initial condition in studying the dynamics of motion of black holes by numerical methods [e.g., see Smarr *et al.* (1976), Smarr (1979)].

The above-described method of constructing the spacetime geometry for a system of black holes at the moment of time symmetry can be extended to the case of rotating black holes [Bowen and York (1980), Bowen *et al.* (1984), Kulkarni (1984)]. On the possibility of equilibrium stationary axially symmetric configurations composed of rotating black holes, see, for example, Oohara and Sato (1981), Kihara and Tomimatsu (1982), Tomimatsu and Kihara (1982), Sato (1983), Tomimatsu (1983), Yamazaki (1983a, b), Bicak and Hoenselaers (1985), and the bibliography therein.

A number of recent papers treat the wormhole-type solution (see Figure 74) in a different context. Morris and Thorne (1987) and Morris, Thorne, and Yurtsever (1988) were able to show that in all likelihood, the laws of physics do not forbid the creation of a wormhole traversable from one mouth to another in any direction. Furthermore, the metric inside the 'handle' of the wormhole may be static and unaltered while the distance between the two mouths in the external space can change in response to their relative motion. Having brought the mouths, connected by a short handle, to a sufficiently large separation, one may hopefully enploy this construct for rapid interstellar travel [Morris and Thorne (1987)]. Furthermore, if one of the mouths is made to recede very rapidly away from the other, then to reverse the motion and return to it, it can be shown that closed time-like lines are formed in spacetime. This construct is a 'time machine' that allows travelling from future into the past [Morris, Thorne, and Yurtsever (1988)]. Novikov (1988) suggested a different version of time machine. Novikov and Thorne (1988) gave an analysis of causality problems entailed by the time machine.

Chapter 9

Quantum Effects in Black Holes. Particle Production

9.1. Role Played by Quantum Effects in Black-Hole Physics

So far, our description has completely ignored quantum specifics of interaction of matter and physical fields with black holes. Quantum effects are, indeed, insignificant for black holes of about solar mass (and greater). However, for a small-mass black hole these effects cannot be ignored; in fact, they qualitatively change the black hole evolution. They are also likely to become important for those regions inside a black hole where the classical theory predicts a spacetime singularity.

In view of the current quantum concepts, the physical vacuum (i.e. the state without real particles) is a quite complex entity. Virtual (short-lived) particles are constantly created, interact with one another, and are annihilated in this vacuum. The vacuum is stable in the absence of external fields, that is, the processes here never produce real (long-lived) particles. In an external field, some virtual particles may acquire sufficient energy for becoming real. The result is the effect of quantum creation of particles from vacuum by an external field.

The probability of particle creation in a static external field can be evaluated as follows. Let Γ denote the field strength of the field and g denote the charge of the particles that are created. According to the uncertainty relation, the lifetime of a pair of virtual particles with energy mc^2 is of order $\hbar/mc^2$. Over this period, the particles move at a velocity not exceeding c and thus recede to a characteristic separation of $l_0 \sim \hbar/mc$. The probability of finding a pair of such a particles at a larger distance l, is proportional to $\exp(-l/l_0)$. The same exponential enters the expression for the creation probability of a real particle pair of energy mc^2 if the separation l is such that the work $g\Gamma l$ done over this distance by the field equals mc^2. As a result, the probability w of particle creation in a field of strength Γ is described by an expression of the type

$$w = A \exp(-\beta m^2 c^3/\hbar g\Gamma), \tag{9.1.1}$$

where the constant β (a dimensionless constant of order unity) and the preexponential factor A depend on subtler characteristics of the field.

The production of electron-position pairs in a high-strength external electric field is a well-known example of the creation of particles in an external field. The following expression derived by Schwinger (1951) holds for the rate of particle production by a uniform electric field in unit volume per unit time:

$$\frac{dN}{dt\,dV} = \frac{e^2E^2}{\pi^2\hbar^2 c}\sum_{n=1}^{\infty}\frac{1}{n^2}\exp(-\pi m^2c^3n/eE\hbar), \tag{9.1.2}$$

where E is the electric field strength. It is readily seen that if the field strength is much less than the critical value $E_{cr} = m^2c^3/e\hbar$, this formula agrees with (9.1.1), the numerical coefficient β being equal to π.

Markov and Frolov (1970) pointed to an important role of quantum effects in black-hole physics. Their paper demonstrated that quantum creation of pairs of charged particles in the field of a charged black hole reduces its electric charge practically to zero. If the potential on the black hole surface, Q/r_+, is sufficiently high for pair production ($eQ/r_+ > mc^2$, where m is the electron mass and e its charge), and the charge of the black hole exceeds $\hbar c/e$, the conditions under which the uniform field approximation is valid are met; hence, Schwinger's relation (9.1.2) can be used to estimate the rate of production of charged particles by the field of a charged black hole.

A similar phenomenon of quantum production of particles [see Zel'dovich (1971), Starobinsky (1973), Misner (1972), Unruh (1974)] takes place in the gravitational field of a rotating black hole. Recall that superradiation, discussed in the preceding chapter, is a purely classical phenomenon. This aspect manifests itself, for instance, in the independence of the enhancement coefficient on Planck's constant. As any other classical process, superradiance can be described in quantum terms. In these terms, the phenomenon consists in an increase in the number of quanta in the reflected wave as compared with that in the incident wave. Indeed, in classical terms, the energy of a wave of a fixed frequency is proportional to the square of its amplitude; in quantum terms, it is proportional to the number of quanta. Correspondingly, the enhanced amplitude of the wave of constant frequency signifies the increased number of quanta of the field.

A quantum analogue can be found for the classical phenomenon of superradiance: spontaneous creation of particles from the vacuum in the gravitational field of a rotating black hole. Since only the mean value of the field is zero in the physical vacuum while the fields fluctuate around the zero value, the amplitude of those quantum fluctuations that met the enhancement (superradiance) condition grows continuously; this growth manifests itself in the creation of real quanta of the field.

The effect of the production of quanta in the field of a rotating black hole can be described in a somewhat different manner that brings out more clearly the role of the ergosphere. For the creation of a real particle that is ejected from a black hole without violating the law of energy conservation, it is necessary for the second particle of the virtual pair to gain negative energy. This becomes possible if this particle is in the ergosphere and possesses a certain value of angular momentum.

The work necessary for the transformation of a virtual particle into a real one is done by the gravitational field of the black hole. The real particles leaving the black hole necessarily possess angular momentum whose direction coincides with that of the black hole. The result is a flux of particles outside the rotating black hole which carries away its energy and angular momentum. The characteristic frequency of this radiation is of order of the angular velocity Ω^H of the black-hole rotation, and the total energy and momentum fluxes are

$$-\frac{dE}{dt} \sim \hbar\,(\Omega^H)^2, \tag{9.1.3}$$

$$-\frac{dJ}{dt} \sim \hbar\,\Omega^H, \tag{9.1.4}$$

The maximum angular velocity for a given mass M, $\Omega^H = c^3/2GM$, is achieved for $J = GM^2/c$, where the black hole is extremal. The rates of energy and angular momentum dissipation by a rotating black hole of mass M due to spontaneous production of particles in its field do not exceed the following values:[1]

$$-\frac{dE}{dt} \sim \hbar\frac{c^6}{G^2M^2} \sim 10^{-17}\frac{\text{erg}}{s}\left(\frac{M_\odot}{M}\right)^2, \tag{9.1.5}$$

$$-\frac{dJ}{dt} \sim \hbar\frac{c^3}{GM} \sim 10^{-22}\text{erg}\left(\frac{M_\odot}{M}\right). \tag{9.1.6}$$

These estimates show that the effect is appreciable only for black holes of small mass (much smaller than the solar mass). Note that the above formulas are valid for the production of massless particles (photons, neutrinos, gravitons); massive particles are born at much lower rates.

If a black hole simultaneously possesses an electric charge Q and angular momentum J, particle production results in a reduction of both angular momentum and charge. If the energy of the newborn particle is ε, angular momentum j, and charge e, these parameters satisfy the inequality implied by condition (8.1.23) for superradiative modes:

$$\varepsilon \leq \Omega^H j + \Phi^H e, \tag{9.1.7}$$

where Ω^H is the angular velocity and Φ^H is the electric potential of the black hole.

As the energy, angular momentum, and charge carried away by the created particles satisfy the same restriction (9.1.7) valid for the radiation parameters in superradiance, it is readily verified that the black hole area does not decrease in this process (see Section 8.1). This result shows that the irreducible mass of the black hole does not decrease in such processes either. Particles are created at the expense of the electrostatics energy or rotational energy stored in the black hole.

A very important discovery that considerably changed our understanding of the role played by quantum effects in black hole physics was made by Hawking (1974,

1975). The discovery consisted in proving that quantum creation of particles takes place in neutral nonrotating black holes as well, and that black hole creates and emits particles as if it were a black body heated to the temperature

$$\theta = \hbar\kappa/2\pi ck, \tag{9.1.8}$$

where k is the Boltzmann constant and κ is the surface gravity of the black hole that characterizes the 'strength' of the gravitational field close to its surface (for a Schwarzschild black hole, $\kappa = c^4/4\,GM$).[2]

The result obtained by Hawking admits the following interpretation. Any particle outside the Schwarzschild black hole has positive energy; hence, the quantum process of particle production in the field of this hole must 'create' one of the particles under the horizon. Recall that the Killing vector field $\xi^\mu\partial_\mu = \partial_t$ is space-like under the horizon and the energy $\varepsilon = -\,\xi^\mu p_\mu$ of a particle with momentum p_μ ($p^\mu p_\mu < 0$) is not sign-definite there. General expression (9.1.1) can be used to crudely estimate the probability of this underbarrier process. However, the gravitational interaction has an essential specific feature implied by its tensor nature and, ultimately, by the equivalence principle. Namely, the mass m of a system, related to the total energy ε of the system by the formula $\varepsilon = mc^2$, acts as the gravitational charge of the system, characterizing its gravitational interaction. The probability of the particle creation in a static gravitational field is thus estimated by formula (9.1.1) after setting $g = m$ and $\Gamma = \kappa$. As a result, we find that the probability of particle creation in the black hole field is

$$w \sim \exp(-\beta\varepsilon c/\hbar\kappa) = \exp\left(-\frac{\beta}{2\pi}\frac{\varepsilon}{k\theta}\right). \tag{9.1.9}$$

Although the above arguments are far from rigorous, they nevertheless lead to the correct expression derived by Hawking (the numerical factor β is found to equal 2π).

In the process of the Hawking radiation, a black hole loses mass, so that its surface area decreases. In the general case of a black hole having charge and angular momentum, this process accompanies the processes that remove angular momentum and electric charge.

The result of the application of an external field to the vacuum goes beyond the effect of particle production. In fact, even those virtual particles that do not gain sufficient energy for becoming real and hence, ultimately disappear, nevertheless undergo the forces due to the external field during their short lifetime and thus move differently from the case of no external field. As a result, the contribution of such virtual particles to various local physical observables (e.g., to the mean value of the energy-momentum tensor $\langle T_{\mu\nu}\rangle$) depends on the strength and other characteristics of the external field. In other words, the change in the vacuum expectation values of local observables in an external field in comparison with the original values in the absence of the field (this difference is what can be measured) is a functional of the external field. This effect of the dependence of local observables on the external field is called *vacuum polarization*. The effect may also take place even in the case when for some reason the external field does not create particles.

The distinction between real and virtual particles has a precise meaning in the absence of an external field. However, it ceases to be unambiguous in the region of spacetime where the external field is strong. This ambiguity leads to a difficulty in defining a particle in strong gravitational fields [for a discussion of this problem, see, e.g., Birrel and Davies (1982)]. As a result, it is not always possible to separate the contributions of real and virtual particles to the mean values of local observables, or to give a precise answer to the question about the place of creation of a particle. Uncertainties arising here are manifestations of the general uncertainty relations inherent in quantum mechanics.

One of the manifestations of the vacuum polarization effect is a change in the equations describing the mean expectation value $\langle\Phi\rangle$ of a physical field Φ produced by an external source J. The field Φ of the source J changes the state of virtual vacuum particles interacting with this field. The resulting additional quantum polarization corrections in the equation for $\langle\Phi\rangle$ take into account the back-reaction of the state of virtual particles to the original field Φ. The quantum process of the creation and annihilation of virtual pairs being random, the 'instantaneous' value of the field Φ does not coincide with the mean expectation value $\langle\Phi\rangle$; the field undergoes *quantum fluctuations*. Hence, the description of the field in terms of its mean expectation values has a limited range of applicability. This description is acceptable in the situations in which quantum fluctuations are small in comparison with the mean expectation value of the field.

These general remarks on the possible manifestations of the quantum nature of physical fields and particles are completely relevant when quantum effects in black holes are considered. In this case, the role of the external source producing the field is played by the massive body whose collapse results in the formation of the black hole.

The following simple arguments give a qualitative estimate of the contribution of fluctuation effects in black holes. Assume that a fluctuation of the metric occurred in a spacetime region of characteristic size l, so that the value of g deviated from the expectation value $\langle g\rangle$by δg. The curvature in this region changes by a quantity of order $\delta g/(l^2\langle g\rangle)$, and the value of action S for the gravitational field correspondingly changes by

$$\delta S \sim \frac{\delta g}{\langle g\rangle} l^2 \frac{c^2}{G}. \tag{9.1.10}$$

The probability of such quantum fluctuation is appreciable only if $\delta S \sim h$. Therefore, the following estimate is obtained for the value of the metric fluctuation $\delta g/\langle g\rangle$ in a spacetime region of size l:

$$\frac{\delta g}{\langle g\rangle} \sim \frac{l_{\rm Pl}^2}{l^2}, \tag{9.1.11}$$

where $l_{\rm Pl} = (\hbar G/c^3)^{1/2} \approx 1.6 \times 10^{-33}$ cm is the Planck length. The matrix fluctuations are thus generally insignificant on a large scale, although they reach unity on the

Planck scale. The mean field approximation is definitely justified for the description of black holes with a mass much greater than the Planck mass $m_{Pl} \sim 10^{-5} g$. It can be expected [York (1983)] that the quantum gravitational fluctuations outlined above lead to an effect resembling quantum 'trembling' of the event horizon. The amplitude of 'trembling' δr_g of the gravitation radius has the following form for a spherically symmetric black hole, as we find from (9.1.11):

$$\delta r_g \sim l_{Pl}^2 / r_g. \tag{9.1.12}$$

It is interesting to note that although this quantity is small for the black holes we discuss here (those with $M \gg m_{Pl}$), the very existence of this effect changes qualitatively the idealized classical description of the collapse and of the falling of a body into a black hole (for a distant observer). The formally infinite expression

$$\Delta l \sim \int_{r_g}^{R} \frac{dr}{1 - r_g/r} \tag{9.1.13}$$

for the duration of these processes by the clock of a distant observer must now be replaced with a finite quantity $\Delta t \sim r_g \ln (r_g/l_{Pl})$, as a result of the substitution $r_g \to r_g + \delta r_g$ in the lower bound of integration.[3]

Now we will discuss the possible role of polarization effects. It can be shown [De Witt (1965)] that after quantum polarization effects have been taken into account, the field $\langle g \rangle$ is described by an equation arising as a result of variation of the quantity

$$W[\langle g \rangle\ ] = \frac{1}{16\pi} \int L_{\rm eff} (\langle g \rangle)\, d^4 v, \tag{9.1.14}$$

called the *effective action*. In the absence of quantum effects (for $\hbar$=0), the effective action coincides with Einstein's action. An expansion of the type

$$L_{\rm eff} = R + \hbar\ L^{(1)} + \cdots \tag{9.1.15}$$

can be used in general case to analyze $L_{\rm eff}$. It can be expected (using, e.g., arguments based on dimensional analysis) that in the approximation lowest in $\hbar$ the quantum corrections to $L_{\rm eff}$ are of order l_{Pl}^2/L^4, where L is the characteristic curvature radius of spacetime. The first term in expansion being of order $R \sim 1/L^2$, the quantum polarization effects may substantially change Einstein's equations when curvature is comparable with $1/l_{Pl}^2$.

In the case of the Schwarzschild metric, this condition is met for $r \sim r_1 = l_{Pl}(r_g/l_{Pl})^{1/3}$ inside the event horizon, provided the black hole mass M is greater than the Planck mass. Hence, quantum effects appreciably change the mean field $\langle g \rangle$ in comparison with the classical solution for $M \gg m_{Pl}$ when $r < r_1$, but the role of these effects is insignificant at the boundary and outside the black hole.

If all terms in the equations of the mean field $\langle g \rangle$, except the Einstein term (corresponding to $\hbar$ = 0), are carried to the right-hand side, the corresponding expression in the right-hand side (it is nonzero only if $\hbar \neq 0$) can be interpreted as

the vacuum expectation value $\langle T^{\nu}_{\mu}\rangle$ of the energy-momentum tensor of those physical fields whose contribution was included into the effective action. The characteristic values of the components $\langle T^{\nu}_{\mu}\rangle$ at the event horizon of a Schwarzschild black hole are of order $\hbar c/r_g^4$. Note that if $M \gg m_{Pl}$, $\langle T^{\nu}_{\mu}\rangle$ affects the external geometry of the black hole only slightly. Nevertheless this small change can result an substantial qualitative changes of the global properties of the solutions describing the black hole if time for this is sufficiently long. Thus, the negative energy flux across the event horizon of an evaporating black hole, accompanying its Hawking radiation, ultimately results in reducing the event horizon to a Planck scale (perhaps even to its complete disappearance). It is readily verified that the expectation value $\langle T^{\nu}_{\mu}\rangle$ for the energy flux across the event horizon, accompanying the Hawking effect, is again of order $\hbar c/r_g^4$.

These arguments show that as long as the mass of a black hole is much greater than the Planck mass and we are dealing with time intervals much shorter than the characteristic time of black hole evaporation, the back-reaction of the created matter and vacuum polarization are negligible and the solutions of classical Einstein's equations can be used to describe the black hole geometry. The effects of quantum fluctuations are also negligible in this approximation. A self-consistent description of the evolution of the external geometry of a black hole can be obtained in the framework of an approximation in which quantum polarization corrections are calculated in a given fixed metric.

This chapter describes the main results concerning the creation of particles in a given gravitational field of a stationary black hole; the polarization effects are analyzed in the next chapter.

9.2. Quantum Creation of Particles in an External Field. General Theory

To prove the results mentioned in the preceding section in connection with the quantum production of particles in black holes, and to extract more detailed information on these quantum phenomena, we will need a sufficiently elaborated mathematical apparatus of quantum field theory in curved spacetime.

Formally, the problems of particle creation in black holes is a particular case of a more general problem of particle creation in arbitrary external fields. The standard scheme of constructing the theory is as follows. One chooses an external field in such a way that the field in the distant past and in the distant future vanish. The concepts of particle and vacuum admit unambiguous definitions in these in- and out-regions. Thus the lowest-energy state of the system is usually chosen as the vacuum. The system undergoes the evolution in an external field and particles are created in the original vacuum state. As a result, the evolution of the state corresponding to the in-vacuum produces a state that is not identical to the out-vacuum state. The operator relating the in- and out-states contains complete information on the processes of particle creation, their scattering, and annihilation in the external field; it is known as the S-matrix.

The problem of particle creation in black holes has two very important aspects that necessitate a modification of the standard scheme. First, it is impossible to 'switch off' in a natural manner the gravitational field of the newborn black hole in the future, although the gravitational field in the past (before the collapse began) can be considered weak in a physically realistic formulation of the problem of collapse and all the states in the in-region can be determined. A decrease in the black hole mass enhances the surface gravity, instead of reducing it and, hence, enhances the intensity of radiation. This is why the 'switching off' of the gravitational field of a black hole by formally reducing its mass does not yield the desired result.

The second point is more important. Namely, a distant observer can record the state of only those particles that escape from the hole. The created particles that sink into the black hole are 'invisible' to this observer. When the results of observations outside a black hole are described, one has to average over these particle states. In other words, the observer outside the black hole invariably deals with only a part of the total quantum system. In accordance with the general principles of quantum mechanics, black-hole radiation is described by a *density matrix*, even if the initial state (before the formation of the black hole) was purely quantum-mechanical. Note that the necessary averaging involves just those states that correspond to the 'particles' not leaving the region of the strong field. It is for these particles that the very concept of 'particle' is poorly defined, since the field of the black hole cannot be 'switched off'. Fortunately, the result of averaging, describing the state of radiation of the black hole, is independent of the arbitrariness in the choice of the method of describing these 'invisible' states.

To summarize, we remark that the problem of interest in this section, namely, the calculation of the characteristics of quantum radiation of a black hole, is naturally broken into two stages: calculation of the S-matrix operator and averaging it over a part of the out-states that correspond to 'invisible' particles. The general formalism of constructing the S-matrix for problems in an external field will be given in this section and the problem of density-matrix calculation for the black-hole radiation is left for the next section.

We give below a brief description of the scheme of constructing a quantum theory of free Bose fields in a given (not necessarily gravitational) field.[4]

The general expression for the action describing a system of real Bose fields $\phi_A(x)(A = 1, ..., M)$ interacting with an arbitrary, given external field $g_Y(x)(Y = 1, ..., Q)$ is written in the form

$$S[\phi] = -\frac{1}{2}\int [\phi_{A,\mu} P^{AB\mu\nu} \phi_{B,\nu} + \phi_A N^{AB\mu} \phi_{B,\mu} + \phi_A T^{AB} \phi_B]\, d^4x, \tag{9.2.1}$$

where $P^{AB\mu\nu} = P^{(AB)(\mu\nu)}$, $N^{AB\mu} = N^{[AB]\mu}$, $T^{AB} = T^{(AB)}$ are real functions of the external field g_Y and its derivatives. Recall that summation is carried out over repeated indices (including the indices A and B). The variation of this action in dynamic variables ϕ_A results in the following field equations:

$$D^{AB}\phi_B \equiv [P^{AB\mu\nu}\partial_\mu\partial_\nu - (N^{AB\mu} - P^{AB\mu\nu}{}_{,\nu})\partial_\mu - (T^{AB} + \frac{1}{2}N^{AB\mu}{}_{,\mu})]\phi_B = 0. \tag{9.2.2}$$

An arbitrary pair of functions ϕ_A^1 and ϕ_A^2 obeys the relation

$$\phi_A^2 D^{AB} \phi_B^1 - \phi_A^1 D^{AB} \phi_B^2 = [B^\mu(\phi^1, \phi^2)]_{,\mu}, \tag{9.2.3a}$$

where

$$B^\mu(\phi^1, \phi^2) = \phi_A^2 P^{AB\mu\nu} \phi_{B,\nu}^1 - \phi_A^1 P^{AB\mu\nu} \phi_{B,\nu}^2 + \phi_A^1 N^{AB\mu} \phi_B^2. \tag{9.2.3b}$$

Hence, it can be shown, using Stokes' theorem, (A.32), [5] that the expression

$$B(\phi^1, \phi^2) = \int_\Sigma B^\mu(\phi^1, \phi^2)\, d\sigma_\mu \tag{9.2.4}$$

calculated for an arbitrary pair of solutions ϕ^1 and ϕ^2 of Equation (9.2.2) is independent of the choice of the total Cauchy surface. The antisymmetric bilinear form B, defined on the space of solutions of Equations (9.2.2), will be called the *canonical form* corresponding to this equation.

In quantum theory, the field $\hat{\phi}_A(x)$ is treated as the operator solution of Equation (9.2.2). The canonical commutation relations imposed on this operator are formulated in a standard manner. Let

$$\hat{\pi}^A(x) = \frac{\delta S}{\delta \phi_{A,0}(x)} = -P^{AB0\nu} \phi_{B,\nu} + \frac{1}{2} N^{AB0} \phi_B$$

be the momentum of the field $\hat{\phi}_A$; then

$$[\hat{\phi}_A(\mathbf{x}, x^0), \hat{\phi}_B(\mathbf{x}', x^0)] = 0,$$

$$[\hat{\pi}^A(\mathbf{x}, x^0), \hat{\pi}^B(\mathbf{x}', x^0)] = 0,$$

$$[\hat{\phi}_A(\mathbf{x}, x^0), \hat{\pi}^B(\mathbf{x}', x^0)] = i\delta_A^B \delta^3(\mathbf{x}, \mathbf{x}'). \tag{9.2.5}$$

Simple check shows that canonical commutation relations (9.2.5) are completely equivalent to the relations [6]

$$[B(\phi^1, \hat{\phi}), B(\phi^2, \hat{\phi})] = iB(\phi^1, \phi^2) \tag{9.2.6}$$

provided the equality is satisfied for an arbitrary pair ϕ_A^1 and ϕ_A^2 of classical solutions of system (9.2.2).

In order to give the definition of particle, it is often convenient to consider the set of complex solutions satisfying the same equations (9.2.2) and the same boundary conditions as the field ϕ_A, and to choose in this space a basis, that is, a complete system of linearly independent solutions. It is convenient to demand that this basis consist of complex-conjugate solutions $u_A^i(x)$, $\bar{u}_A^i(x)$ satisfying the following *normalization conditions*:

$$B(u^i, u^j) = B(\bar{u}^i, \bar{u}^j) = 0, \quad B(u^i, \bar{u}^j) = i\delta^{ij}. \tag{9.2.7}$$

Here the indices i, j, ... enumerate basic solutions. Each solution of equations (9.2.2), satisfying the imposed boundary conditions, admits a decomposition over this basis. Thus, for the field operator $\hat{\phi}_A$, we can write

$$\hat{\phi}_A(x) = \sum_i [\hat{a}_i \overset{i}{u}_A(x) + \hat{a}_i^* \overset{i}{\bar{u}}_A(x)], \tag{9.2.8}$$

where

$$\hat{a}_i = iB(\overset{i}{\bar{u}}, \hat{\phi}), \quad \hat{a}_i^* = -iB(\overset{i}{u}, \hat{\phi}). \tag{9.2.9}$$

If the field $\hat{\phi}_A$ is Hermitian ($\hat{\phi}_A^* = \hat{\phi}_A$)), then $\hat{a}_i^* = (\hat{a}_i)^*$. The constant operators $\hat{a}_i^*$ and $\hat{a}_i$, called the *particle creation* and *annihilation operators* in a state with the wave function $\overset{i}{u}_A(x)$, satisfy the following commutation relations:

$$[\hat{a}_i, \hat{a}_j] = [\hat{a}_i^*, \hat{a}_j^*] = 0, \; [\hat{a}_i, \hat{a}_j^*] = \delta_{ij}. \tag{9.2.10}$$

This is readily verified if the relation (9.2.6) and normalization conditions (9.2.7) are used.

The vacuum state $|0\rangle$ corresponding to a given choice of a basis is defined by the condition

$$\hat{a}_i |0\rangle = 0. \tag{9.2.11}$$

A state $|i_1, ..., i_n\rangle$ with n particles whose wave functions are $\overset{i_1}{u}_A(x), ..., \overset{i_n}{u}_A(x)$ is obtained from the vacuum by applying to it the appropriate number of creation operators:

$$|i_1, ..., i_n\rangle = \hat{a}_{i_1}^* \dots \hat{a}_{i_n}^* |0\rangle . \tag{9.2.12}$$

These multiparticle basis states constitute the eigenstates of the operator $\hat{n}_i = \hat{a}_i^* \hat{a}_i$ of the number of particles in the ith modes,

$$\hat{n}_i |i_1, ..., i_n\rangle = n_i |i_1, ..., i_n\rangle , \quad n_i = \sum_{k=1}^{n} \delta_{i_k} \tag{9.2.13}$$

and satisfying the following orthonormalization and completeness conditions:

$$\langle i_1, ..., i_n | j_1, ..., j_m \rangle = 0 \;\; \text{if } n \neq m$$

$$\langle i_1, ..., i_n | j_1, ..., j_n \rangle = \sum_{\substack{\text{over all permutations} \\ (i_1, ..., i_n)}} \delta_{i_1 j_1} \dots \delta_{i_n j_n} , \tag{9.2.14}$$

$$\hat{I} = |0\rangle \langle 0| + \sum_{n=1}^{\infty} |i_1, ..., i_n\rangle \frac{1}{n!} \langle i_1, ..., i_n | , \tag{9.2.15}$$

where $\hat{I}$ is the identity operator and summation is carried out over all sets $(i_1, ..., i_n)$.

Obviously, the choice of the basis $(u^i_A, \bar{u}^i_A)$ and of the related definition of a 'particle' is far from unambiguous. The formal construction given above becomes physically meaningful only if we succeed is describing clearly a set of attributes sufficient for distinguishing between the vacuum (or one-particle) state and all other allowed quantum states of the system. In the long run, this problem reduces to describing a detector capable of recording the particles. According to quantum theory of measurements, this instrument is described by an Hermitian operator whose eigenvectors are the states corresponding to a certain number of particles recorded by this instrument.

All necessary concepts can be defined in the asymptotic in- and out-regions in the standard scheme of the theory, where the external field is 'switched off' in the distant past and distant future. In each of these regions, one has to operate with the free Bose field theory in flat spacetime where the conserved operators of energy, $\hat{P}_0$, and momentum, $\hat{P}_i$, corresponding to the translations along the time (x^0) and spatial (x^i) coordinates in Minkowski space, are defined unambiguously:

$$\left[\hat{P}_a, \hat{\phi}\right] = \frac{1}{i}\xi^\mu_a \partial_\mu \hat{\phi}. \tag{9.2.16}$$

Here $\xi^\mu_a = \delta^\mu_a$ are Killing vector fields generating the respective translations. The vacuum state in each of the asymptotic regions is defined as the lowest eigenstate of the energy operator $\hat{P}_0$. This choice unambiguously corresponds to choosing functions with positive frequency with respect to the time coordinate x^0 as basis functions:

$$\xi^\mu_0 \partial_\mu u^i_A(x) = -i\omega_i u^i_A. \tag{9.2.17}$$

In order to distinguish between two bases consisting of the solutions $(u^i_A, \bar{u}^i_A)$, for which relations (9.2.1) act as the asymptotic boundary conditions in the future (out-basis) and in the past (in-basis), we equip the basis functions with a subscript *out* and *in*, respectively. Likewise, we make use of these additional indices in order to distinguish between the quantities defined via these bases. For instance,

$$\hat{\phi}_A = \sum_i (\hat{a}_{\text{in},A} u^i_{\text{in},A} + \hat{a}^*_{\text{in},i} \bar{u}^i_{\text{in},A})$$

$$= \sum_i (\hat{a}_{\text{out},i} u^i_{\text{out},A} + \hat{a}^*_{\text{out},i} \bar{u}^i_{\text{out},A}), \tag{9.2.18}$$

$$\hat{a}_{\text{in},i}|0;\text{in}\rangle = 0, \hat{a}_{\text{out},i}|0;\text{out}\rangle = 0. \tag{9.2.19}$$

In general, the in- and out-bases do not coincide because the in- and out-basis functions satisfy different boundary conditions. The coefficients of the matrices A^{ij} and B^{ij} that relate the in- and out-bases,

$$u^{i}_{\text{out},A} = \sum_j (A^{ij} u^{j}_{\text{in},A} + B^{ij} \bar{u}^{j}_{\text{in},A}), \tag{9.2.20}$$

are known as *Bogoliubov transformation coefficients*. Using the normalization conditions (9.2.7), we find

$$A^{ij} = iB(\bar{u}^{j}_{\text{in}}, u^{i}_{\text{out}}), \quad B^{ij} = -iB(u^{j}_{\text{in}}, u^{i}_{\text{out}}). \tag{9.2.21}$$

The following matrix notation is convenient for contracting the expressions:

$$\hat{b}_{\substack{\text{in}\\ \text{out}}} = (\hat{a}_{\substack{\text{in},i\\ \text{out}}}, \hat{a}^{*}_{\substack{\text{in},i\\ \text{out}}}),$$

$$v_{\substack{\text{in}\\ \text{out}}} = \begin{pmatrix} u^{i}_{\substack{\text{in}\\ \text{out}}} \\ \bar{u}^{i}_{\substack{\text{in}\\ \text{out}}} \end{pmatrix}, \quad C = \begin{pmatrix} A^{ij} & B^{ij} \\ \bar{B}^{ij} & \bar{A}^{ij} \end{pmatrix}. \tag{9.2.22}$$

Using this notation, we obtain

$$v_{\text{out}} = C v_{\text{in}}, \quad \hat{b}_{\text{in}} = \hat{b}_{\text{out}} C, \quad \hat{\phi} = \hat{b}_{\text{in}} v_{\text{in}} = \hat{b}_{\text{out}} v_{\text{out}}, \tag{9.2.23}$$

$$\hat{b}_{\text{out}} = \hat{b}_{\text{in}} C^{-1}, \quad v_{\text{in}} = C^{-1} v_{\text{out}}.$$

Relations (9.2.21) yield the following expression for the coefficients of matrix C^{-1} inverse to C:

$$C^{-1} = \begin{pmatrix} A^{+} & -B' \\ -B^{+} & A' \end{pmatrix}. \tag{9.2.24}$$

Primes in the above formulas denote transposition, and + stands for Hermitian conjugation of matrices: $(\ )^{+} = \overline{(\ )}'$. The conditions $CC^{-1} = C^{-1}C = I$ imply that the following equalities are true:

$$AA^{+} - BB^{+} = I, \quad A^{+}A - B'\bar{B} = I,$$

$$(A^{-1}B)' = A^{-1}B, \quad (\bar{B}A^{-1})' = \bar{B}A^{-1} \tag{9.2.25}$$

(here I is the identity matrix).

The S-matrix operator relating the in- and out-states is defined by the relations

$$\hat{b}_{\text{in}} \hat{S} = \hat{S} \hat{b}_{\text{out}}. \tag{9.2.26}$$

It can be verified that this operator is unitary ($\hat{S}\hat{S}^{+} = \hat{I}$), has the property

$$\hat{S} | i_1, \ldots, i_n; \text{out} \rangle = | i_1, \ldots, i_n; \text{in} \rangle , \tag{9.2.27}$$

and admits the following representation:

$$\hat{S} = \sum_n |i_1, ..., i_n; \text{in}\rangle \ \frac{1}{n!} \langle i_1, ..., i_n; \text{out}|. \tag{9.2.28}$$

If we substitute $\hat{b}_{\text{in}} = \hat{b}_{\text{out}} C$ into (9.2.26) and solve the resulting equation, the operator $\hat{S}$ can be expressed in terms of $\hat{b}_{\text{out}}$. The corresponding solution admits the following representation [Berezin (1965), De Witt (1965)]:

$$\hat{S} = e^{iW_0} :\exp(\tfrac{1}{2}\, \hat{a}'_{\text{out}} \Lambda \hat{a}_{\text{out}} + \hat{a}^{+}_{\text{out}} (M - I) \hat{a}_{\text{out}} + \tfrac{1}{2}\, \hat{a}^{+}_{\text{out}} V \hat{a}^{*}_{\text{out}}):, \tag{9.2.29}$$

where : : denotes the operation of normal ordering [7] with respect to the out-operators, $\hat{a}^{*}_{\text{out}} = (\hat{a}^{+}_{\text{out}})'$, and

$$\Lambda = A^{-1} B, \quad V = -\bar{B} A^{-1}, \quad M = A^{-1\,'},$$

$$e^{iW_0} = \theta[\det(A^{+} A)]^{-1/4}, \quad |\theta| = 1 \tag{9.2.30}$$

(here $\hat{a}'_{\text{out}} \Lambda \hat{a}_{\text{out}} \equiv \Sigma_{i,j}\ \hat{a}_{\text{out},i} \Lambda^{ij}\ \hat{a}_{\text{out},j}$, etc.).

Equalities (9.2.25) imply that the matrices Λ and V are symmetric:

$$\Lambda' = \Lambda, \quad V' = V. \tag{9.2.31}$$

The result (9.2.29) is fundamental for the quantum theory in the external field that we are discusing here. Namely, it states that it is possible to calculate explicitly the S-matrix operator containing exhaustive information on quantum effects of creation, scattering, and absorption of particles in the external field, provided we know the Bogoliubov transformation coefficients determined by solving classical equations (9.2.2). It can be shown that the matrices V, M and Λ in (9.2.9) are related directly to probability amplitudes of the elementary processes of creation, scattering, and annihilation in the external field:

$$\langle i, j; \text{out} \,|\, 0; \text{in}\rangle = e^{iW_0} V^{ij},$$

$$\langle i; \text{out} \,|\, j; \text{in}\rangle = e^{iW_0} M^{ij}, \tag{9.2.32}$$

$$\langle 0; \text{out} \,|\, i, j; \text{in}\rangle = e^{iW_0} \Lambda^{ij}.$$

9.3. Averaging over 'Nonobservable' States. Density Matrix

Let us discuss in more detail the features that distinguish the problem of particle creation in black holes from the general problem in the external field considered in the preceding Section. We have already mentioned the possibility of dividing the set of out-states into classes is a characteristic of processes involving black holes; we refer to the representative of these classes as 'visible' and 'invisible'. The first class consists of states corresponding to particles moving away from the black hole, and the second class, to particles falling into it. (By convention, we subsume bound

states under the second class.) To make this partitioning explicit, we choose to replace the index i enumerating the out-states with α for enumerating 'visible' states and a for enumerating 'invisible' ones. We also introduce the notation

$$\hat{\beta}_\alpha \equiv \hat{a}_{\text{out},\,\alpha}, \quad \hat{b}_a \equiv \hat{a}_{\text{out},\,a},$$

$$|\alpha_1, \ldots, \alpha_k; \beta\rangle \; |a_1, \ldots, a_m; b\rangle \equiv \hat{\beta}^*_{\alpha_1} \ldots \hat{\beta}^*_{\alpha_k} \hat{b}^*_{a_1} \ldots \hat{b}^*_{a_m} |0; \text{out}\rangle . \tag{9.3.1}$$

An arbitrary vector in the space of out-states $|\Psi\rangle$ admits the following decomposition:

$$|\Psi\rangle = \sum_{k,\,m} \sum_{\substack{\alpha_1,\ldots,\alpha_k \\ a_1,\ldots,a_m}} \Psi_{\alpha_1,\ldots,\alpha_k;\, a_1,\ldots,a_m} |\alpha_1, \ldots, \alpha_k; \beta\rangle \; |a_1, \ldots, a_m; b\rangle . \tag{9.3.2}$$

For the expectation value $\langle\Psi| \hat{F} |\Psi\rangle$ of an arbitrary operator F that depends only on the 'visible' states $\hat{F} = F(\hat{\beta}^*_\alpha, \hat{\beta}_\alpha)$, we find

$$\langle\Psi| \hat{F} |\Psi\rangle = \sum_{\substack{k,\,m \\ k',\,m'}} \sum_{\substack{\alpha_1,\ldots,\alpha_k \\ \dot{\alpha}_1,\ldots,\dot{\alpha}_{k'} \\ a_1,\ldots,a_m \\ \dot{a}_1,\ldots,\dot{a}_{m'}}} \bar{\Psi}_{\dot{\alpha}_1,\ldots,\dot{\alpha}_{k'};\, \dot{a}_1,\ldots,\dot{a}_{m'}} \Psi_{\alpha_1,\ldots,\alpha_k;\, a_1,\ldots,a_m} \times$$

$$\times \langle\dot{\alpha}_1,\ldots,\dot{\alpha}_k ;\beta| \langle\dot{a}_1,\ldots,\dot{a}_m ;b| F(\hat{\beta}^*_\alpha, \hat{\beta}_\alpha) |a_1,\ldots,a_m;b\rangle \; |\alpha_1,\ldots,\alpha_k; \beta\rangle. \tag{9.3.3}$$

Since the operator $\hat{F}$ is independent of $\hat{b}_a$ and $\hat{b}^*_a$ and the states $|a_1,\ldots,a_m;b\rangle$ satisfy the normalization conditions

$$\langle\dot{a}_1,\ldots,\dot{a}_{m'};b \mid a_1,\ldots,a_m;b\rangle = \delta_{mm'} \sum_{\substack{\text{over all} \\ \text{permutations} \\ (a_1,\ldots,a_m)}} \delta_{\dot{a}_1 a_1} \ldots \delta_{\dot{a}_{m'} a_m}, \tag{9.3.4}$$

we can rewrite (9.3.3) in the form

$$\langle\Psi| F |\Psi\rangle = \sum_{k,k'} \sum_{\substack{\alpha_1,\ldots,\alpha_k, \\ \dot{\alpha}_1,\ldots,\dot{\alpha}_{k'}}} R_{\alpha_1,\ldots,\alpha_k;\, \dot{\alpha}_1,\ldots,\dot{\alpha}_{k'}} \langle\dot{\alpha}_1,\ldots,\dot{\alpha}_{k'};\beta| \hat{F} |\alpha_1,\ldots,\alpha_k;\beta\rangle$$

$$= \mathrm{Sp}_\beta(\hat{\rho}\hat{F}), \tag{9.3.5}$$

where

$$R_{\alpha_1,\ldots,\alpha_k;\dot{\alpha}_1,\ldots,\dot{\alpha}_{k'}} = \sum_m m! \sum_{a_1,\ldots,a_m} \bar{\Psi}_{\dot{\alpha}_1,\ldots,\dot{\alpha}_{k'};a_1,\ldots,a_m} \Psi_{\alpha_1,\ldots,\alpha_k;a_1,\ldots,a_m}, \tag{9.3.6}$$

$$\hat{\rho} = \sum_{k,k'} \sum_{\substack{\alpha_1,\ldots,\alpha_k \\ \alpha'_1,\ldots,\alpha'_{k'}}} |\alpha_1,\ldots,\alpha_k;\beta\rangle \, R_{\alpha_1,\ldots,\alpha_k;\alpha'_1,\ldots,\alpha'_{k'}} \langle\alpha'_1,\ldots,\alpha'_{k'};\beta| \tag{9.3.7}$$

and Sp_β denotes the operation of trace calculation in the state space of 'visible' particles.

We have to emphasize that the density matrix introduced above is *independent* of the method of defining the concept of particle for 'invisible' states. We have already mentioned using the example of transformations relating in- and out-bases, that the operator $\hat{S}$ corresponding to this transformation is unitary: $\hat{S}^+ = \hat{S}^{-1}$. Obviously, this property also holds for similar general-type canonical transformations. A change of basis in the subspace of solutions corresponding to 'invisible' particles is described in terms of a unitary operator $\hat{U}$ having the property

$$\hat{U}|\alpha_1,\ldots,\alpha_k;\beta\rangle\,|a_1,\ldots,a_m;b\rangle$$

$$\equiv \sum_{\substack{m' \\ a'_1,\ldots,a'_{m'}}} U_{a_1,\ldots,a_m;a'_1,\ldots,a'_{m'}} |\alpha_1,\ldots,\alpha_k;\beta\rangle\,|a'_1,\ldots,a'_{m'};b\rangle\,, \tag{9.3.8}$$

$$\sum_{\substack{m' \\ a'_1,\ldots,a'_{m'}}} U_{a_1,\ldots,a_m;a'_1,\ldots,a'_{m'}} \bar{U}_{a''_1,\ldots,a''_{m''};a'_1,\ldots,a'_{m'}}$$

$$= \delta_{mm''}\delta_{a_1a''_1}\ldots\delta_{a_ma''_{m''}}\,. \tag{9.3.9}$$

The coefficients $\Psi_{\alpha_1,\ldots,\alpha_k;a_1,\ldots,a_m}$ of expansion (9.3.2) are transformed as follows:

$$\Psi'_{\alpha_1,\ldots,\alpha_k;a'_1,\ldots,a'_{m'}}$$

$$= \sum_{\substack{m \\ a_1,\ldots,a_m}} \Psi_{\alpha_1,\ldots,\alpha_k;a_1,\ldots,a_m} U_{a_1,\ldots,a_m;a'_1,\ldots,a'_{m'}}\,, \tag{9.3.10}$$

while the coefficients of the matrix $R_{\alpha_1,\ldots,\alpha_k;\alpha'_1,\ldots,\alpha'_{k'}}$ remain unaltered as a result of the unitarity condition (9.3.9).

It is readily verified, using relation (9.3.5) for the identity operator $\hat{F} = \hat{I}$, that the density matrix $\hat{\rho}$ for the normalized state $|\Psi\rangle(\langle\Psi|\Psi\rangle = 1)$ satisfies the normalization condition $Sp_\beta(\hat{\rho}) = 1$.

9.4. Density Matrix and Generating Functional for Quantum Effects in Black Holes

Now that the general formal scheme has been outlined, we turn to its application to the description of quantum effects in black holes. For the sake of simplicity, we limit the analysis to the theory of massless neutral scalar field ϕ in the spacetime of a rotating black hole. The case of massless fields is the most important because, on one hand, massless fields give the main contribution to the quantum radiation of black holes, and on the other hand, this analysis serves as a good approximation in describing the creation of massive particles in the case when the Hawking temperature of the black hole is much higher than the rest energy of these particles, so that the ultrarelativistic approximation is valid for their description. We will later return to discussing the effects of spin, mass, and charge of particles on the processes of their creation in black holes.

Figure 75 shows the Penrose diagram for the spacetime of a rotating black hole produced in the collapse of a massive body. We assume that the coordinate of Bondi's advanced conformal time v is chosen in such a way that a light signal sent from $\mathcal{J}^-$ at the moment $v = 0$ reaches the point $r = 0$ exactly at the moment at which horizon appears (see Figure 75). Since a black hole becomes almost stationary rather soon after formation, we assume that wave packets emitted from $\mathcal{J}^-$ at some moment of advanced time $v = v_1$ propagate all the time in a metric identical to the Kerr metric. In order to construct the basis wave functions, we make use of the expansion of the solutions to the wave equation

$$\phi\Box = 0 \tag{9.4.1}$$

in the Kerr metric in spheroidal wave functions

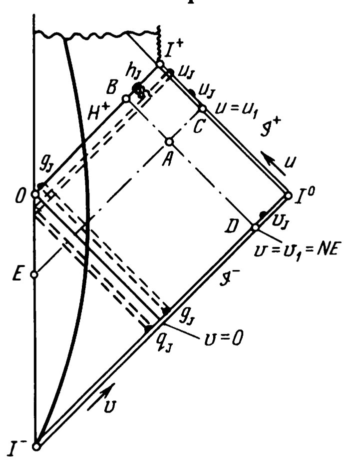


Fig. 75. Penrose diagram of the spacetime of a rotating black hole produced by the collapse of a massive body.

$$Y_{lm}(\theta, \phi) = \frac{e^{im\phi}}{\sqrt{2\pi}} S_l^m(\cos\theta), \tag{9.4.2}$$

where S_l^m are defined as eigenfunctions, bounded on the interval $[-1,1]$, of the operator

$$\left\{\frac{d}{dz}\left[(1-z^2)\frac{d}{dz}\right] - \omega^2 a^2 (1-z^2) - \frac{m^2}{1-z^2}\right\} S_l^m(z) = -\lambda_l^m S_l^m(z), \tag{9.4.3}$$

satisfying the normalization conditions

$$\int_{-1}^{1} S_l^m(z) S_{l'}^m(z)\, dz = \delta_{ll'}. \tag{9.4.4}$$

We denote by $v_{\omega lm}$the solution to Equation (9.4.1) that has the following property: Its image $V_{\omega lm}$ on $\mathcal{I}^-$, defined as

$$V_{\omega lm}(v, \theta, \phi) = \lim_{\substack{r\to\infty \\ v,\,\theta,\,\phi=\text{const}}} [r v_{\omega lm}(r, v, \theta, \phi)]$$

has the form

$$V_{\omega lm} = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega v} Y_{lm}(\theta, \phi), \tag{9.4.5}$$

where v is the coordinate of Bondi's advanced conformal time (see Section 5.1). In what follows, it is convenient to consider as the basis functions not $v_{\omega lm}$ but wave-packet-type solutions constructed of $v_{\omega lm}$. To achieve this, we fix a real number E ($0 < E \ll 1$) and introduce the notation (for $j \geq 0$)

$$v_{jnlm} = E^{-1/2} \int_{jE}^{(j+1)E} e^{2\pi i n \omega/E} v_{\omega lm}\, d\omega. \tag{9.4.6}$$

Hereafter we denote the collective index $jnlm$ by a single letter α. The wave packets v_α on $\mathcal{I}^-$ have frequencies in the interval from jE to $(j+1)E$ and reach a maximum close to the moment $v = 2\pi n/E$ of advanced time; the packet width is $\Delta v \sim 2\pi/E$.

Note that for the massless field theory discussed here, the surface $\mathcal{I}^-$ in the Penrose space plays the role of the total Cauchy surface, and the canonical form B for a pair ϕ^1, ϕ^2 of solutions to Equation (9.4.1),

$$B(\phi^1, \phi^2) = \int_\Sigma (\phi^2 \phi^1_{,\nu} - \phi^1 \phi^2_{,\nu}) g^{\nu\mu} d\sigma_\mu \tag{9.4.7}$$

admits the following presentation:

$$B(\phi^1, \phi^2) = \int_{\mathcal{I}^-} (\Phi^1\, \partial_v \Phi^2 - \Phi^2\, \partial_v \Phi^1)\, dv d\omega, \tag{9.4.8}$$

Using (9.4.2), (9.4.4) – (9.4.6), and (9.4.8), it is easily shown that wave packets v_α satisfy the normalization conditions

$$B(v_\alpha, v_{\alpha'}) = B(\bar{v}_\alpha, \bar{v}_{\alpha'}) = 0, \quad B(v_\alpha, \bar{v}_{\alpha'}) = i\,\delta_{\alpha\alpha'} \tag{9.4.9}$$

and form, together with $\bar{v}_\alpha$, a complete system on $\mathcal{J}^-$. Here, and hereafter, we use the notation

$$\delta_{\alpha\alpha'} \equiv \delta_{jj'}\,\delta_{nn'}\,\delta_{ll'}\,\delta_{mm'}. \tag{9.4.10}$$

This set of functions, $(v_\alpha, \bar{v}_\alpha)$, can be chosen for the in-basis. The in-vacuum state $|0; \text{in}\rangle$, corresponding to this choice, is singled out by the condition of vanishing energy flux from $\mathcal{J}^-$ onto the black hole.

Replacing Bondi's advanced conformal time v in (9.4.5) by the retarded time u we obtain functions on $\mathcal{J}^+$ that we denote by $U_{\omega lm}$. Using now a transformation similar to (9.4.6), we construct on $\mathcal{J}^+$ the wave packets u_α. The formation of the event horizon H^+ implies that the solution to wave equation (9.4.1) is not completely defined when the image of the function on $\mathcal{J}^+$ has been specified. Additional conditions required to make the solution unique can be described as the values assumed by the solution at the horizon H^+. Let us define the wave packet u_α as solution (9.4.1) that vanishes at the event horizon and has an image U_α on $\mathcal{J}^+$. Obviously, these packets satisfy the normalization conditions similar to (9.4.9). If the solutions $(u_\alpha, \bar{u}_\alpha)$ are complemented to a complete system by arbitrary functions $(h_\alpha, \bar{h}_\alpha)$ whose images vanish on $\mathcal{J}^+$ and form a complete normalized system of functions on H^+, then the system $(u_\alpha, \bar{u}_\alpha, h_\alpha, \bar{h}_\alpha)$ can be chosen as the out-basis. It is convenient to choose for h_α the solutions defined as follows. Let $\tilde{h}_{\omega lm}$ be the solutions that vanish on $\mathcal{J}^+$ and assume on H^+ the values [in coordinates (4.4.1)]

$$\tilde{h}_{\omega lm}\big|_{H^+} = \frac{1}{\sqrt{4\pi|\omega - m\Omega^H|(r_+^2 + a^2)}}\, e^{-i\omega\tilde{v}}\, Y_{lm}(\theta, \tilde{\phi}), \tag{9.4.11}$$

and let $\tilde{h}_\alpha$ be wave packets (9.4.6) constructed for these solutions. Let us define h_α using the relation

$$h_\alpha = \theta(\sigma_\alpha)\tilde{h}_\alpha + \theta(-\sigma_\alpha)\bar{\tilde{h}}_\alpha, \quad \sigma_\alpha = \text{sign}(\omega_j - m\Omega^H). \tag{9.4.12}$$

It can be shown that the functions h_α satisfy the following normalization conditions:

$$B(h_\alpha, h_{\alpha'}) = B(\bar{h}_\alpha, \bar{h}_{\alpha'}) = 0, \quad B(h_\alpha, \bar{h}_{\alpha'}) = i\,\delta_{\alpha\alpha'}. \tag{9.4.13}$$

Wave equation (9.4.1) allows total separation of variables in the spacetime of an eternal black hole described by a Kerr metric; hence, the wave functions introduced above are related by the following formula:

$$v_{\omega lm} = R_{\omega lm}\, u_{\omega lm} + T_{\omega lm}\, \tilde{h}_{\omega lm}. \tag{9.4.14}$$

We refer to $R_{\omega lm}$ and $T_{\omega lm}$ as the *coefficients of relection* and *absorption* of a wave $v_{\omega lm}$ by a black hole. The value of $|R_{\omega lm}|^2$ equals the ratio of the energy of the scattered wave to that of the incident one. This ratio exceeds unity for waves satisfying the superradiance condition. If we choose the parameter E of packets to be sufficiently small, and if $R_{\omega lm}$ and $T_{\omega lm}$ are smooth functions of frequency ω, a similar decomposition can also be written for wave packets:

$$v_\alpha = R_\alpha u_\alpha + T_\alpha \tilde{h}_\alpha . \tag{9.4.15}$$

Note that wave packets ($v_\alpha \equiv v_{jnlm}$) with sufficiently high values of n ($\geq N \equiv v_1/E$) in the spacetime of a black hole produced by collapse propagate in a metric that is almost identical to the Kerr metric (see Figure 75). Hence, relation (9.4.15) holds for them as well. The following relation is implied by the conditions of normalization for the functions v_α, u_α, and h_α:

$$|R_\alpha|^2 + \sigma_\alpha |T_\alpha|^2 = 1 . \tag{9.4.16}$$

This expression implies, among other things, that $|R_\alpha|^2 > 1$ for those modes that satisfy the superradiance condition $\sigma_\alpha < 0$.

The next step is to find the Bogoliubov transformation coefficients that relate the constructed in- and out-bases. The problem is greatly simplified if we make use of the following techniques suggested by Wald (1975). In what follows we consider, without expressly mentioning it, the wave packets whose subscript α satisfies the condition $n \geq N$, so that the packets with these subscripts satisfy condition (9.4.15). Let us define wave packets $\tilde{q}_\alpha$ such that they are orthogonal to v_α,

$$B(\tilde{q}_\alpha, v_\alpha) = B(\tilde{q}_\alpha, \bar{v}_\alpha) = 0, \tag{9.4.17}$$

and admit the decomposition

$$\tilde{q}_\alpha = t_\alpha u_\alpha + r_\alpha \tilde{h}_\alpha . \tag{9.4.18}$$

By q_α we denote the wave packet related to $\tilde{q}_\alpha$ by the formula

$$q_\alpha = \theta(\sigma_\alpha)\tilde{q}_\alpha + \theta(-\sigma_\alpha)\bar{\tilde{q}}_\alpha \tag{9.4.19}$$

and normalized by the condition

$$B(q_\alpha, \bar{q}_{\alpha'}) = i\delta_{\alpha\alpha'} . \tag{9.4.20}$$

The conditions of orthogonality and normalization imply that the following relations hold, in addition to (9.4.16):

$$|r_\alpha|^2 + \sigma_\alpha |t_\alpha|^2 = 1, \quad t_\alpha \bar{R}_\alpha + \sigma_\alpha r_\alpha \bar{T}_\alpha = 0 . \tag{9.4.21}$$

The symmetry property of the Kerr metric under the transformation $t \to -t$, $\phi \to -\phi$ imply the equality [see, e.g. Unruh (1974)]

$$T_\alpha = \sigma_\alpha t_\alpha . \tag{9.4.22}$$

Using relations (9.4.15), (9.4.16), (9.4.18), (9.4.21), and (9.4.22), we find that

$$u_\alpha = \bar{T}_\alpha \tilde{q}_\alpha + \bar{R}_\alpha v_\alpha. \tag{9.4.23}$$

This expression shows that if we trace the evolution of a packet u_α into the past, some part of it $(\bar{R}_\alpha v_\alpha)$ reaches $\mathscr{I}^-$ after being scattered by the stationary field of the black hole, and the remaining part $(\bar{T}_\alpha \tilde{q}_\alpha)$ is transmitted through the collapsing body at the instant of time that precedes the formation of the event horizon, and reaches $\mathscr{I}^-$ at $v < 0$. Hawking proved (1975) that the geometrical optics approximation can be used for describing the second part and that in this approximation the wave packet $\tilde{Q}_\alpha$ (the image of q_α on $\mathscr{I}^-$) is obtained from the function

$$\tilde{Q}_{\omega l m} = \frac{1}{\sqrt{4\pi\tilde{\omega}}} e^{i\tilde{\omega}\ln(-v)/\kappa} Y_{lm}(\theta, \phi)\theta(-v) \tag{9.4.24}$$

by applying transformation (9.4.6). Here $\tilde{\omega} = \omega - m\Omega^H$ and $\theta(x)$ is a step function that is nonzero and equals 1 for $x > 0$.

This result is obtained by considering the behavior of the constant-phase surface for the wave $u_{\omega l m}$. In the geometrical optics approximation, this is a null surface. The equation $\tilde{U} =$ const, where $\tilde{U}$ is the retarded Kerr coordinate, gives a description of this surface outside the collapsing body where the spacetime geometry is well approximated by the Kerr metric. The generators of the surface are then null geodesics $\tilde{U} =$ const, $\tilde{\phi}_+ =$ const, $\theta =$ const (see Section 4.4). If extended into the past, these geodesics pass through the collapsing matter and emerge on $\mathscr{I}^-$ at a point with coordinates $v, \phi, \theta (v < 0)$. It can be shown [Hawking (1975)], that the following relations, connecting $\tilde{U}$, $\tilde{\phi}_+$ with v, ϕ holds as $\tilde{U} \to \infty$:

$$\tilde{U} = -\kappa^{-1} \ln(-v), \quad \tilde{\phi}_+ = -\Omega^H \kappa^{-1} \ln(-v) + \phi.$$

If we also recall that the coordinates $\tilde{U}$, $\tilde{\phi}_+$ coincide on $\mathscr{I}^+$ with u, ϕ, we arrive at expression (9.4.24) for $Q_{\omega l m}$. This approximation is the more accurate, the later the moment of time u at which the packet u_α reaches $\mathscr{I}^+$. We assume that the number N was chosen to be sufficiently large, so that the approximation provides sufficiently high accuracy.

Let us introduce another family of solutions g_α, having defined them by specifying the images G_α on $\mathscr{I}^-$:

$$G_\alpha(v, \theta, \phi) = \bar{Q}_\alpha(-v, \theta, \phi). \tag{9.4.25}$$

Let us show that linear combinations p_α and n_α of the functions g_α and q_α,

$$p_\alpha = c_\alpha g_\alpha + s_\alpha \bar{q}_\alpha, \quad n_\alpha = c_\alpha q_\alpha + s_\alpha \bar{g}_\alpha, \tag{9.4.26}$$

$$s_\alpha = (w_\alpha^{-2\sigma_\alpha} - 1)^{-1/2}, \quad c_\alpha = (1 - w_\alpha^{2\sigma_\alpha})^{-1/2},$$

$$w_\alpha = \mathrm{e}^{-\pi(\omega_j - m\Omega^H)/\kappa}, \tag{9.4.27}$$

have images on $\mathcal{I}^-$ that are of positive frequency with respect to the advanced time v.

To prove this statement, it is sufficient to note that the functions p_α and n_α are obtained by applying transformation (9.4.6) to solutions that manifest the following dependence on v on the infinity $\mathcal{I}^-(-\infty < \tilde{\alpha} < \infty)$:

$$F_{\omega l m}(v) = \theta(v)\,\mathrm{e}^{-i\tilde{\omega}\ln v/\kappa} + \theta(-v)\,\mathrm{e}^{-\pi\tilde{\omega}/\kappa}\mathrm{e}^{-i\tilde{\omega}\ln(-v)/\kappa}. \tag{9.4.28}$$

On the other hand,

$$\int_{-\infty}^{0} \mathrm{e}^{-ipv}\,\mathrm{e}^{-i\tilde{\omega}\ln(-v)/\kappa}\,dv$$
$$= -\mathrm{e}^{\pi\tilde{\omega}/\kappa}\int_{0}^{\infty} \mathrm{e}^{-ipv}\,\mathrm{e}^{-i\tilde{\omega}\ln v/\kappa}\,dv, \quad \text{for all } p > 0. \tag{9.4.29}$$

This equality is readily verified by distorting the integration contour in the right-hand side of (9.4.29) in the lower half-plane. Therefore, we obtain for $p > 0$ that

$$\int_{-\infty}^{\infty} \mathrm{e}^{-ipv} F_{\omega l m}(v)\,dv = 0, \tag{9.4.30}$$

which implies that p_α and n_α satisfy the condition of positive frequency functions.

We choose for in-basis positive-frequency solutions the sets of functions

$$W_\alpha^{\text{in}} = \begin{pmatrix} v_\alpha \\ p_\alpha \\ n_\alpha \end{pmatrix}, \quad \alpha = jnlm, \tag{9.4.31}$$

for n greater than the already chosen N, and supplement them with an arbitrary set of positive-frequency functions defined on $\mathcal{I}^-$ to a complete orthonormalized basis. Likewise, we form the out-basis by supplementing a set of functions

$$W_\alpha^{\text{out}} = \begin{pmatrix} u_\alpha \\ h_\alpha \\ g_\alpha \end{pmatrix}, \quad n > N, \tag{9.4.32}$$

to a complete orthonormalized system. This choice of bases, suggested by Wald (1975), is convenient because this choice factorizes the Bogoliubov transformation matrices, so that we have

$$W_\alpha^{\text{in}} = A_\alpha^{+} W_\alpha^{\text{out}} - B_\alpha^{'} \overline{W}_\alpha^{\text{out}}. \tag{9.4.33}$$

Transformation matrices A_α and B_α that relate the sets of in-basis (W_α^{in}) and out-basis (W_α^{out}) functions are easily determined from relations (9.4.15), (9.4.18), and (9.4.26); they are

$$A_\alpha = \begin{pmatrix} \bar{R}_\alpha & s_\alpha \bar{t}_\alpha & 0 \\ 0 & 0 & c_\alpha r_\alpha \\ 0 & c_\alpha & 0 \end{pmatrix}, \quad B_\alpha = \begin{pmatrix} 0 & 0 & -c_\alpha \bar{t}_\alpha \\ -T_\alpha & -s_\alpha r_\alpha & 0 \\ 0 & 0 & s_\alpha \end{pmatrix}, \quad \sigma_\alpha < 0,$$

$$A_\alpha = \begin{pmatrix} \bar{R}_\alpha & 0 & c_\alpha \bar{t}_\alpha \\ \bar{T}_\alpha & 0 & c_\alpha \bar{r}_\alpha \\ 0 & c_\alpha & 0 \end{pmatrix}, \quad B_\alpha = \begin{pmatrix} 0 & -s_\alpha \bar{t}_\alpha & 0 \\ 0 & -s_\alpha \bar{r}_\alpha & 0 \\ 0 & 0 & -s_\alpha \end{pmatrix}, \quad \sigma_\alpha > 0. \tag{9.4.34}$$

The above-described method of transforming to the Wald bases thus yields an explicit expression for the Bogoliubov transformation coefficients that determine the relation between in- and out-basis functions at large values of $n \geq N$. The expression for the S-matrix operator can be obtained using the general formulas (9.2.29) and (9.2.30).

The next step is the calculation of the density matrix that describes the black hole radiation. We denote by

$$\hat{\beta}_\alpha = iB(\bar{u}_\alpha, \hat{\varphi}), \quad \hat{\beta}^*_\alpha = -iB(u_\alpha, \hat{\varphi})$$

the operators of creation and annihilation of particles in the state u_α. Let us take up the calculation of expectation values of the type $\langle 0;\text{ in} | F(\hat{\beta}^*_\alpha, \hat{\beta}_\alpha) | 0;\text{ in}\rangle$. Using expression (9.2.29) and (9.2.30) for the S-matrix operator and substituting $\hat{S}|0;\text{ out}\rangle$ for $|0;\text{ in}\rangle$, we can find the expansion coefficients (9.3.2) and then calculate the sought density matrix.

In the general case, the density matrix $\hat{\rho}$ (it describes the observables on $\mathcal{J}^+$ and appears in the averaging over the states h_α) depends on the finer details of black hole formation. However, if we are interested in the values of observables on $\mathcal{J}^+$ only at sufficiently late moments of retarded time ($u > u_1$), these details are found to be unimportant, so that the values assumed by these observables depend only on the parameters of the resulting stationary black hole. This statement is readily verified if we consider the density matrix $\hat{\rho}_N$ obtained from ρ by additionally averaging over all states on $\mathcal{J}^+$ except u_α with $n \geq N$. To obtain an explicit expression for $\hat{\rho}_N$, it is sufficient to know the Bogoliubov transformation coefficients A_α and B_α, calculated above, for $n \geq N$. We omit the calculational details [they can be found in Frolov's papers (1983a, 1986)] and give only the final result,

$$\hat{\rho}_N = \prod_{\substack{\alpha \\ n \geq N}} \hat{\rho}_\alpha,$$

$$\hat{\rho}_\alpha = Q_\alpha :\exp[-Q_\alpha \hat{\beta}^*_\alpha \hat{\beta}_\alpha]: = Q_\alpha \exp[\ln(1 - Q_\alpha) \hat{\beta}^*_\alpha \hat{\beta}_\alpha], \tag{9.4.35}$$

where $Q_\alpha = (1 - w_\alpha^2)(1 - w_\alpha^2 |R_\alpha|^2)^{-1}$ and : : denotes the operation of normal ordering with respect to the operators $\hat{\beta}_\alpha$ and $\hat{\beta}^*_\alpha$ (The last equality takes into account the familiar relation $:\exp \tau \hat{\beta}^* \hat{\beta} : = \exp [\ln (1 + \tau) \hat{\beta}^* \hat{\beta}]$.)

Expression (9.4.35) for nonrotating black holes was derived by Hawking. Let $|n_\alpha>$ be state with n_α particles in the mode α. Then

$$\hat{\rho}_\alpha = \sum_{n_\alpha, n'_\alpha} \rho_{n_\alpha n'_\alpha} |n_\alpha\rangle \langle n'_\alpha|. \tag{9.4.36a}$$

For a nonrotating black hole,

$$\rho_{n_\alpha n'_\alpha} = \delta_{n_\alpha n'_\alpha} (\Gamma_\alpha)^{n_\alpha} (e^{\omega/\theta} - \varepsilon)^{\varepsilon} (e^{\omega/\theta} - \varepsilon + \varepsilon\Gamma_\alpha)^{-n_\alpha + \varepsilon}, \tag{9.4.36b}$$

where $\Gamma_\alpha = |T_\alpha|^2$ and $\varepsilon = 1$. A similar expression is valid for $\hat{\rho}_\alpha$ in the case of fermions as well, but ε then equals -1 and n_α can assume the values 0 and 1 [Wald (1975), Hawking (1976b)]. If we neglect the scattering by the gravitational field ($R_\alpha = 0$), the above expression becomes

$$\hat{\rho}_N = e^{-\hat{\mathcal{H}}_0/\theta}, \tag{9.4.37a}$$

where $\hat{\mathcal{H}}_0$ is the free Hamiltonian describing particles escaping to $\mathcal{J}^+$:

$$\hat{\mathcal{H}}_0 = \sum_{\substack{\alpha \\ n \ge N}} \omega \hat{\beta}^*_\alpha \hat{\beta}_\alpha, \tag{9.4.37b}$$

and

$$\theta = \kappa/2\pi \tag{9.4.38}$$

is the Hawking temperature of the black hole.

Expression (9.4.35) for the density matrix $\hat{\rho}_N$ makes it possible to calculate the expectation values of the observables, $\hat{F} = \hat{F}(\hat{\beta}^*, \hat{\beta})$ on $\mathcal{J}^+$,

$$\langle 0; \text{in}|F(\hat{\beta}^*_\alpha, \hat{\beta}_\alpha)|0; \text{in}\rangle = \text{Sp}_\beta(\hat{\rho}_N \hat{F}). \tag{9.4.39}$$

The following method, well known in quantum field theory and quantum statistics, permits substantial simplification of the calculation of (9.4.39)-type expressions. For $\hat{F}$, we choose the operator

$$\hat{K}[\psi, \bar{\psi}] = \exp\left(\sum_{\substack{\alpha \\ n \ge N}} \psi_\alpha \hat{\beta}^*_\alpha\right) \exp\left(\sum_{\substack{\alpha \\ n \ge N}} \bar{\psi}_\alpha \hat{\beta}_\alpha\right)$$

$$\equiv :\exp\left[\sum_{\substack{\alpha \\ n \ge N}} (\psi_\alpha \hat{\beta}^*_\alpha + \bar{\psi}_\alpha \hat{\beta}_\alpha)\right]:. \tag{9.4.40}$$

Note that if the operator $\hat{F}$ in (9.4.39) is given in the normal form,

$$\hat{F} = F(\hat{\beta}^*_\alpha, \hat{\beta}_\alpha)$$

$$= \sum_{m,m'} \sum_{\substack{\alpha_1,\dots,\alpha_m \\ \alpha'_1,\dots,\alpha'_{m'}}} F_{\alpha_1,\dots,\alpha_m;\alpha'_1,\dots,\alpha'_{m'}} \hat{\beta}^*_{\alpha_1} \dots \hat{\beta}^*_{\alpha_m} \hat{\beta}_{\alpha'_1} \dots \hat{\beta}_{\alpha'_{m'}}, \tag{9.4.41}$$

it can be written in the form

$$F(\hat{\beta}^*_\alpha, \hat{\beta}_\alpha) = \left[F\left(\frac{\partial}{\partial \psi_\alpha}, \frac{\partial}{\partial \bar{\psi}_\alpha} \right) K(\psi, \bar{\psi}) \right]_{\psi = \bar{\psi} = 0}, \tag{9.4.42}$$

where

$$F\left(\frac{\partial}{\partial \psi_\alpha}, \frac{\partial}{\partial \bar{\psi}_\alpha} \right)$$

$$= \sum_{m,m'} \sum_{\substack{\alpha_1,\dots,\alpha_m \\ \alpha'_1,\dots,\alpha'_{m'}}} F_{\alpha_1,\dots,\alpha_m;\alpha'_1,\dots,\alpha'_{m'}} \times$$

$$\times \frac{\partial^m}{\partial \psi_{\alpha_1} \dots \partial \psi_{\alpha_m}} \frac{\partial^{m'}}{\partial \bar{\psi}_{\alpha'_1} \dots \partial \bar{\psi}_{\alpha'_{m'}}}. \tag{9.4.43}$$

We denote

$$Z[\psi, \bar{\psi}] = \mathrm{Sp}_\beta[\hat{\rho}_N \hat{K}(\psi, \bar{\psi})], \tag{9.4.44}$$

and thus obtain

$$\langle 0; \mathrm{in}| F(\hat{\beta}^*_\alpha, \hat{\beta}_\alpha) |0; \mathrm{in}\rangle = \left\{ F\left(\frac{\partial}{\partial \psi_\alpha}, \frac{\partial}{\partial \bar{\psi}_\alpha} \right) Z[\psi, \bar{\psi}] \right\}_{\psi = \bar{\psi} = 0}. \tag{9.4.45}$$

In other words, instead of each time calculating $\langle \hat{F} \rangle \equiv \mathrm{Sp}_\beta(\hat{\rho}_N \hat{F})$ anew, it is sufficient that we calculate this quantity once for $\hat{F} = \hat{K}[\psi, \bar{\psi}]$ and find the quantity $Z[\psi, \bar{\psi}]$ called the *generating functional*. Now the problem of calculating $\langle \hat{F} \rangle$ reduces to differentiating $Z[\psi, \bar{\psi}]$.

The class of problems solvable by the method described can be considerably widened after a slight modification. First, it is convenient to choose for $\hat{K}$ the following expression:

$$\hat{K}[\psi, \bar{\psi}, \mu] =:\exp\left[-\sum_{\substack{\alpha\\ n\geq N}} (\mu_\alpha \hat{\beta}^*_\alpha \hat{\beta}_\alpha - \psi_\alpha \hat{\beta}^*_\alpha - \bar{\psi}_\alpha \hat{\beta}_\alpha)\right]: . \tag{9.4.46}$$

Since differentiation with respect to μ_α gives $\hat{n}_\alpha = \hat{\beta}^*_\alpha \hat{\beta}_\alpha$, the introduction of a dependence on μ into $\hat{K}$ makes it possible to easily calculate the expectation values of expressions that contain operators of the number of particles $\hat{n}_\alpha$. Second, the introduction of additional variables γ_α, $\bar{\gamma}_\alpha$ into Z yields formulas that allows calculation of expectation values of $\hat{F}$ not only in the vacuum state but also in an arbitrary multiparticle initial state. Indeed, we notice that

$$\langle \Psi | \hat{F} | \Psi \rangle = \mathrm{Sp}(\hat{\rho}_\Psi \hat{F}). \tag{9.4.47}$$

where

$$\hat{\rho}_\Psi = |\Psi\rangle \langle \Psi|. \tag{9.4.48}$$

For instance,

$$\langle 0; \mathrm{in}| \hat{F} |0; \mathrm{in}\rangle = \mathrm{Sp}(\hat{\rho}_0 \hat{F}), \tag{9.4.49}$$

where

$$\hat{\rho}_0 = |0; \mathrm{in}\rangle \langle 0; \mathrm{in}| = :\exp\left(-\sum_\alpha \hat{a}^*_{\mathrm{in},\alpha} \hat{a}_{\mathrm{in},\alpha}\right):. \tag{9.4.50}$$

Here : : denotes the operation of normal ordering with respect to the operators

$$\hat{a}_{\mathrm{in},\alpha} = iB(\bar{\upsilon}_\alpha, \hat{\phi}), \quad \hat{a}^*_{\mathrm{in},\alpha} = -iB(\upsilon_\alpha, \hat{\phi}). \tag{9.4.51}$$

Define the generating functional $Z[\psi, \mu; \gamma]$ by the relation

$$Z[\psi, \mu; \gamma] = \mathrm{Sp}(\hat{\rho}_\gamma \hat{K}[\psi, \bar{\psi}, \mu]), \tag{9.4.52}$$

where

$$\hat{\rho}_\gamma = :\exp\left[-\sum_{\substack{\alpha\\ n\geq N}} \hat{a}^*_{\mathrm{in},\alpha} \hat{a}_{\mathrm{in},\alpha} + \sum_{\substack{\alpha\\ n\geq N}} (\gamma_\alpha \hat{a}^*_{\mathrm{in},\alpha} + \bar{\gamma}_\alpha \hat{a}_{\mathrm{in},\alpha})\right]:. \tag{9.4.53}$$

If $\mu_\alpha = 0$, $\gamma_\alpha = \bar{\gamma}_\alpha = 0$, then $Z[\psi, \mu; \gamma]$ obviously coincides with (9.4.44).

The explicit expression for the generating functional $Z[\psi, \mu; \gamma]$ has the following form [Frolov (1983a, 1986)]:

$$Z[\psi, \mu; \gamma] = \prod_{\substack{\alpha\\ n\geq N}} Z_\alpha[\psi_\alpha, \mu_\alpha; \gamma_\alpha], \tag{9.4.54}$$

where

$$Z_\alpha[\psi_\alpha, \mu_\alpha; \gamma_\alpha] = \frac{Q_\alpha}{C_\alpha}\exp\left(\frac{P_\alpha}{C_\alpha}\right). \tag{9.4.55}$$

Here,

$$Q_\alpha = (1 - w_\alpha^2)(1 - w_\alpha^2 |R_\alpha|^2)^{-1},$$

$$C_\alpha = Q_\alpha + \mu_\alpha - Q_\alpha\mu_\alpha,$$

$$P_\alpha = (C_\alpha - \mu_\alpha Q_\alpha |R_\alpha|^2)\bar{\gamma}_\alpha\gamma_\alpha + Q_\alpha R_\alpha \gamma_\alpha \bar{\psi}_\alpha +$$

$$+ Q_\alpha \bar{R}_\alpha \bar{\gamma}_\alpha \psi_\alpha + (1 - Q_\alpha)\bar{\psi}_\alpha\psi_\alpha, \tag{9.4.56}$$

R_α is the reflection coefficient in (9.4.15), and w_α is defined by equality (9.4.27).

To conclude this section, we will give the general relations that establish the connection of the generating functional $Z[\psi,\mu;\gamma]$ to the main quantities that are of physical interest and characterize quantum effects in black holes.

Let us use the following notation:

$$D_\alpha^{l_\alpha} = \frac{1}{(l_\alpha)!}\frac{\partial^{l_\alpha}}{\partial\gamma_\alpha^{l_\alpha}}\frac{\partial^{l_\alpha}}{\partial\bar{\gamma}_\alpha^{l_\alpha}}, \qquad \Delta_\alpha^{k_\alpha} = \frac{1}{(k_\alpha)!}\frac{\partial^{k_\alpha}}{\partial\psi_\alpha^{k_\alpha}}\frac{\partial^{k_\alpha}}{\partial\bar{\psi}_\alpha^{k_\alpha}}. \tag{9.4.57}$$

(1) Let

$$|\alpha_1', \ldots, \alpha_m'; \text{in}\rangle = \hat{a}^*_{\text{in},\alpha_1'} \ldots \hat{a}^*_{\text{in},\alpha_m'}|0; \text{in}\rangle,$$

$\hat{Q} = Q(\hat{n}_{\alpha_1}, \ldots, \hat{n}_{\alpha_k})$ be a function of the operators $\hat{n}_\alpha \equiv \hat{\beta}^*_\alpha\hat{\beta}_\alpha$ and $\hat{F} = F(\hat{\beta}^*_\alpha, \hat{\beta}_\alpha)$ be an operator given in the (9.4.41)-type normal form. Then

$$\langle\alpha_1', \ldots, \alpha_m'; \text{in}|\, Q(\hat{n}_{\alpha_1}, \ldots, \hat{n}_{\alpha_k})\, |\alpha_1', \ldots, \alpha_m'; \text{in}\rangle$$

$$= \left\{ D^1_{\alpha_1'} \ldots D^1_{\alpha_m'} Q\left(\frac{\partial}{\partial\lambda_{\alpha_1}}, \ldots, \frac{\partial}{\partial\lambda_{\alpha_k}}\right) Z[\psi = 0, \mu = 1 - e^\lambda; \gamma] \right\}_{\lambda=\gamma=0}, \tag{9.4.58}$$

$$\langle\alpha_1', \ldots, \alpha_m'; \text{in}|\, F(\hat{\beta}^*_\alpha, \hat{\beta}_\alpha)\, |\alpha_1', \ldots, \alpha_m'; \text{in}\rangle$$

$$= \left\{ D^1_{\alpha_1'} \ldots D^1_{\alpha_m'} F\left(\frac{\partial}{\partial\psi_\alpha}, \frac{\partial}{\partial\bar{\psi}_\alpha}\right) Z[\psi, \mu = 0; \gamma] \right\}_{\psi=\gamma=0}. \tag{9.4.59}$$

(2) Let $P(k_{\alpha_1},\ldots, k_{\alpha_n} | l_{\alpha_1},\ldots, l_{\alpha_n})$ be the probability that k_{α_1} quanta are emitted by the black hole to the infinity $\mathcal{J}^+$ in the mode $u_{\alpha_1},\ldots$, and k_{α_n} quanta in the mode u_{α_n}, provided there were l_{α_1} quanta on $\mathcal{J}^-$ in the mode $v_{\alpha_1},\ldots$, and l_{α_n} quanta in the mode $v_{\alpha_n}(k_{\alpha_i} \geq 0,\ l_{\alpha_i} \geq 0,\ k_{\alpha_i} + l_{\alpha_i} > 0)$. The following equalities then hold:

$$P(k_{\alpha_1},\ldots, k_{\alpha_n} | l_{\alpha_1},\ldots, l_{\alpha_n}) = P(k_{\alpha_1} | l_{\alpha_1})\ldots P(k_{\alpha_n} | l_{\alpha_n}),$$

$$P(k_\alpha | l_\alpha) = (D_\alpha^{l_\alpha} \Delta_\alpha^{k_\alpha} Z_\alpha[\psi_\alpha, 1; \gamma_\alpha])_{\gamma_\alpha = \psi_\alpha = 0}. \tag{9.4.60}$$

Therefore, the calculation of the expectation values of observables, correlation functions, and probability distributions for processes in the field of a black hole reduces to the operation of differentiation of the generating functional Z defined by relations (9.4.54)–(9.4.56). Note that the generating functional is completely defined if in addition to the surface gravity κ and angular velocity Ω^H of the black hole, we also know the coefficients of reflection, R_α, and absorption, T_α, of wave packets v_α by the black hole. For reasons described above, it is sufficient to know these values for wave packets v_α propagating in the stationary Kerr metric of the created black hole. To find the coefficients R_α and T_α, the one-dimensional scattering problem has to be solved. We can write the solution $v_{\omega lm}$ of Equation (9.4.1) in the Kerr metric in the form

$$v_{\omega lm} = \frac{1}{\sqrt{4\pi\omega}} \frac{1}{\sqrt{r^2 + a^2}} e^{-i\omega t} w_{\omega lm}(r) Y_{lm}(\theta, \phi), \tag{9.4.61}$$

where $w_{\omega lm}$ satisfies the radial equation

$$\frac{d^2}{dr^{*2}} w_{\omega lm} + U_{\omega lm} w_{\omega lm} = 0 \tag{9.4.62}$$

and the following boundary conditions:

$$w_{\omega lm} \propto \begin{cases} e^{-i\omega r^*} + R_{\omega lm} e^{i\omega r^*}, & r^* \to \infty, \\ \left|\dfrac{\omega}{\tilde{\omega}}\right|^{1/2} T_{\omega lm} e^{-i\tilde{\omega} r^*}, & r^* \to -\infty. \end{cases} \tag{9.4.63}$$

Here,

$$\tilde{\omega} = \omega - m\Omega^H, \qquad \frac{dr^*}{dr} = \frac{r^2 + a^2}{r^2 - 2Mr + a^2}, \tag{9.4.64}$$

$$U_{\omega lm} = (\tilde{\omega} - A_{\omega lm})^2 - B_{\omega lm}, \tag{9.4.65}$$

$$A_{\omega l m} = \frac{am}{r^2 + a^2} - \Omega^H m,$$

$$B_{\omega l m} = \Delta \left[\frac{\lambda_l^m - 2a\omega m}{(r^2 + a^2)^2} + \frac{1}{(r^2 + a^2)^{3/2}} \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{r\Delta}{(r^2 + a^2)^{3/2}} \right) \right], \tag{9.4.66}$$

and λ_l^m is the eigenvalue corresponding to spheroidal function S_l^m (9.4.3).

The problem of calculating $R_{\omega l m}$ and $T_{\omega l m}$, and a similar problem for fields with other characteristics (nonzero spin, mass, and charge) have been analyzed in detail in numerous papers. For detailed discussion and explicit formulas for the reflection and absorption coefficients in these cases, see Chandrasekhar (1983) where references to original publications are also given.

The method of the generating functional developed in this section for a scalar massless field can be generalized to the case of higher spins. Explicit expressions for the density matrix and the generating functional for the neutrino ($s = 1/2$), photon ($s = 1$), and graviton ($s = 2$) fields in the spacetime of a rotating black hole and their application to the calculation of observables can be found in Bolashenko and Frolov (1988, 1989).

9.5. Particular Cases

In this section we discuss a number of specific results following from general relations (9.4.58)–(9.4.60).

(a) *Hawking effect.* Let a system occupy the vacuum state before the black hole was formed. After the formation, the black hole becomes a source of radiation. The mean number of particles radiated in the mode α and recorded by a distant observer, is given by the following expression:

$$\langle \hat{n}_\alpha \rangle_0 \equiv \langle 0; \mathrm{in} | \hat{n}_\alpha | 0; \mathrm{in} \rangle$$

$$= \left\{ \frac{\partial}{\partial \lambda_\alpha} Z_\alpha[0, 1 - e^{\lambda_\alpha}; 0] \right\}_{\lambda_\alpha = 0} = \frac{\Gamma_\alpha}{\exp(\tilde{\omega}_\alpha / \theta) - 1}, \tag{9.5.1}$$

where

$$\Gamma_\alpha = \sigma_\alpha |T_\alpha|^2, \quad \tilde{\omega}_\alpha = \omega_\alpha - m_\alpha \Omega^H, \quad \text{and} \quad \theta = \kappa/2\pi$$

is the black-hole temperature. Note that even though the denominator of (9.5.1) vanishes at $\tilde{\omega}_\alpha = 0$, the mean number of particles created in such modes remains finite because of the simultaneously vanishing $|T_\alpha|^2$. It is also clear that the reversal of the sign of the denominator at $\tilde{\omega}_\alpha < 0$ is accompanied by the reversal of sign of σ_α so that, on the whole, expression (9.5.1) is always positive.

As the absorption coefficient of the packet v_α of a stationary black hole is independent of the moment of time $v = 2\pi n/E$ at which this packet was emitted, the number of particles reaching $\mathscr{J}^+$ is independent of retarded time u. In other words, the newborn black hole becomes a source of a stationary flux of radiation. We have already mentioned above that the appearance of a stationary flux can be interpreted as a corollary of a spontaneous process of particle pair creation in a stationary gravitational field close to the event horizon. One of the particles of each created pair sinks into the black hole while the other escapes to infinity.[8]

Another interpretation of the result (9.5.1) is possible, bringing out more clearly the role played by nonstationarity of the gravitational filed in the process of the black hole formation [this interpretation was discussed, e.g., by Gerlach (1976) and Zel'dovich (1976)]. Let us consider the behavior of a wave packet emitted from $\mathscr{J}^-$ at $v < 0$ with a characteristic frequency ω, so that the packet goes through the collapsing body and emerges from it immediately before the event horizon is formed. This packet undergoes a strong red shift and its frequency changes to ω'. However high the frequency ω, a moment $v < 0$ can be found after which $\omega' \lesssim 1/T_g$, where $T_g = r_g/c$ is the characteristic time of the gravitational collapse.

The action of the variable gravitational field at the stage of the gravitational collapse is essentially nonadiabatic for such quanta. The number of quanta ceases to be conserved in nonadiabatic perturbation, with particles being created in the vacuum state. The higher the frequency of zero vacuum oscillations, the closer to the event horizon they must propagate in order to satisfy the condition of nonadiabaticity and create a real quantum, and the later this quantum reaches the distant observer. This process results in black-hole radiation infinitely extended in time because the vacuum generates zero oscillations of arbitrarily high frequencies.

Note that the processes of particle creation by a stationary field and a time-dependent field have substantially different characteristics in the flat space. The problem of particle creation in the black hole is special in that the creation occurs in the neighborhood of the event horizon which is a *light-like surface*. For this reason, the above-described 'stationary' and 'nonstationary' interpretations of this effect are not contradictory but complementary ones.

(b) *Stimulated radiation.* Let m particles in the mode α fall on a black hole from $\mathscr{J}^-$. Then we make use of (9.4.58) and obtain the following expression for the mean number of particles in the state α at $\mathscr{J}^+$:

$$\begin{aligned}\langle \hat{n}_\alpha \rangle_m &= \langle m_\alpha;\, \text{in} | \hat{n}_\alpha | m_\alpha;\, \text{in} \rangle \\ &= \{ D_\alpha^{m_\alpha} [(-1 + Q_\alpha^{-1} + |R_\alpha|^2 \bar{\gamma}_\alpha \gamma_\alpha)\, e^{\bar{\gamma}_\alpha \gamma_\alpha}] \}_{\gamma=0} = \langle \hat{n}_\alpha \rangle_0 + m_\alpha |R_\alpha|^2. \qquad (9.5.2)\end{aligned}$$

The term $\langle \hat{n}_\alpha \rangle_0$ describes the spontaneous creation of particles from the vacuum; it is given by expression (9.5.1). Bekenstein and Meisels (1977) noticed that the resulting expression can be rewritten in the equivalent form

$$\langle \hat{n}_\alpha \rangle_m = A_\alpha^{\downarrow} + B_\alpha^{\downarrow} m_\alpha + (1 - B_\alpha^{\uparrow}) m_\alpha, \qquad (9.5.3)$$

and the quantities

$$A^{\downarrow}_{\alpha} = B^{\downarrow}_{\alpha} = \langle \hat{n}_{\alpha} \rangle_0 = \frac{\sigma_{\alpha}|T_{\alpha}|^2}{\exp(\tilde{\omega}_{\alpha}/\theta) - 1}, \tag{9.5.4}$$

$$B^{\uparrow}_{\alpha} = \frac{\sigma_{\alpha}|T_{\alpha}|^2}{1 - \exp(-\tilde{\omega}_{\alpha}/\theta)} \tag{9.5.5}$$

can be interpreted as analogues of Einstein's coefficients for processes in black holes. The term $A^{\downarrow}_{\alpha}$ describes the spontaneous creation of particles from the vacuum, $B^{\uparrow}_{\alpha}$ acts as the absorption coefficient, and $B^{\downarrow}_{\alpha} m_{\alpha}$ describes the stimulated emission of particles in the mode α.

The quantity $1 - B^{\uparrow}_{\alpha}$ can be interpreted as the probability of scattering of the mode α by the black hole. Using (9.4.16) and (9.5.4), we rewrite the expression for $|R_{\alpha}|^2$ in the form

$$|R_{\alpha}|^2 = (1 - B^{\uparrow}_{\alpha}) + B^{\downarrow}_{\alpha}. \tag{9.5.6}$$

This formula shows that the quantity $|R_{\alpha}|^2$, characterizing the scattering of a particle in mode α by the black hole, is the sum of the probability of scattering of the incoming particle, $1 - B^{\uparrow}_{\alpha}$, and the probability of stimulated emission of radiation in this mode, $B^{\downarrow}_{\alpha}$. The coefficient of the stimulated emission for superradiative modes, $B^{\downarrow}_{\alpha}$ *exceeds* the absorption coefficient $B^{\uparrow}_{\alpha}$ because $B^{\downarrow}_{\alpha} = B^{\uparrow}_{\alpha} \exp(-\tilde{\omega}_{\alpha}/\theta)$, so that $|R_{\alpha}|^2$ takes on values greater than unity.

Formulas (9.5.3)–(9.5.5) show that the black-hole radiation obeys the same laws that the radiation emitted by a hot body. In fact, an essential difference exists. The black-hole temperature is determined by the same parameters (mass and angular momentum) that determine its geometrical size while the temperature of an ordinary body is an independent parameter.

(c) *Scattering of coherent waves.* We have already mentioned that there is a close relationship between the processes of propagation of a classical wave and individual quanta in an external field. Let us consider this relationship in the case of scattering on a black hole in more detail. Consider a classical wave incident on a black hole. From the quantum point of view, this wave can be described as a coherent ensemble of quanta, and characterized by the following coherent state

$$|\gamma_{\beta}; \text{in}\rangle = \exp(-\frac{1}{2}\bar{\gamma}_{\beta}\gamma_{\beta}) \exp(\gamma_{\beta}\hat{a}^{*}_{\text{in},\beta}) |0; \text{in}\rangle . \tag{9.5.7}$$

The normalization condition of this state,

$$\langle \gamma_{\beta}; \text{in} \mid \gamma_{\beta}; \text{in}\rangle = 1, \tag{9.5.8}$$

is implied by the following relation [see, e.g., Klauder and Sudarshan (1968)]:

$$e^{\hat{X}} e^{\hat{Y}} = e^{[\hat{X},\hat{Y}]} e^{\hat{X}} e^{\hat{Y}}, \tag{9.5.9}$$

which holds for arbitrary operators $\hat{X}$ and $\hat{Y}$, such that $[\hat{X},\hat{Y}]$ commutes with $\hat{X}$ and $\hat{Y}$.

Simple checking confirms that

$$\hat{a}_{\text{in},\beta}\exp(\gamma_\beta \hat{a}^*_{\text{in},\beta}) = \exp(\gamma_\beta \hat{a}^*_{\text{in},\beta})\,(\hat{a}_{\text{in},\beta} + \gamma_\beta). \tag{9.5.10}$$

Using this relation, it can be shown that the mean value of the image $\hat{\Phi}_{\text{in}}$ of the field $\hat{\varphi}$ on $\mathcal{J}^-$,

$$\bar{\hat{\Phi}}_{\text{in}} = \sum_\alpha (V_\alpha \hat{a}^*_{\text{in},\alpha} + \bar{V}_\alpha \hat{a}_{\text{in},\alpha}), \tag{9.5.11}$$

is given in the coherent state (9.5.7) by the expression

$$\Phi^{\text{in}}_{\gamma_\alpha} \equiv \langle \gamma_\beta;\ \text{in}|\hat{\Phi}_{\text{in}}|\gamma_\beta;\ \text{in}\rangle = \gamma_\beta \bar{V}_\beta + \bar{\gamma}_\beta V_\beta. \tag{9.5.12}$$

For the image of scattered wave on $\mathcal{J}^+$, Φ_{out}, we have

$$\Phi^{\text{out}}_{\gamma_\beta} \equiv \langle \gamma_\beta;\ \text{in}|\ \hat{\Phi}_{\text{out}}|\gamma_\beta;\ \text{in}\rangle\ . \tag{9.5.13}$$

Note here that the following relations hold:

$$|\gamma_\beta;\ \text{in}\rangle\ \langle \gamma_\beta;\ \text{in}| = e^{-\bar{\gamma}_\beta \gamma_\beta} \hat{\rho}_\gamma, \tag{9.5.14}$$

$$\Phi^{\text{out}}_{\gamma_\beta} = e^{-\bar{\gamma}_\beta \gamma_\beta}\,\text{Sp}(\hat{\rho}_\gamma \hat{\Phi}_{\text{out}}), \tag{9.5.15}$$

$$\hat{\Phi}_{\text{out}} \equiv \sum_\alpha (U_\alpha \hat{\beta}^*_\alpha + \bar{U}_\alpha \hat{\beta}_\alpha), \tag{9.5.16}$$

$$\hat{\Phi}_{\text{out}} = \left\{ \sum_\alpha \left(U_\alpha \frac{\partial}{\partial \psi_\alpha} + \bar{U}_\alpha \frac{\partial}{\partial \bar{\psi}_\alpha} \right) \hat{K}[\psi, \bar{\psi}, 0] \right\}_{\psi = \bar{\psi} = 0}, \tag{9.5.17}$$

where $\hat{\rho}_\gamma$ and $\hat{K}$ are given by relations (9.4.53) and (9.4.46), respectively, with only those terms containing γ_α and $\bar{\gamma}_\alpha$ being nonzero in (9.4.53) whose subscript α coincides with β. These formulas yield the following relation between $\Phi^{\text{out}}_{\gamma_\beta}$ and the functional Z:

$$\Phi^{\text{out}}_{\gamma_\beta} = e^{-\bar{\gamma}_\beta \gamma_\beta} \left\{ \sum_\alpha \left(U_\alpha \frac{\partial}{\partial \psi_\alpha} + \bar{U}_\alpha \frac{\partial}{\partial \bar{\psi}_\alpha} \right) Z[\psi, 0; \gamma] \right\}_{\psi = 0}. \tag{9.5.18}$$

Using relations (9.4.54)–(9.4.56) for Z, we finally obtain

$$\Phi^{\text{out}}_{\gamma_\beta} = \gamma_\beta R_\beta \bar{U}_\beta + \bar{\gamma}_\beta \bar{R}_\beta U_\beta . \tag{9.5.19}$$

In other words, the ratio of squared amplitude of the scattered wave to squared amplitude of the incident wave is $|R_\beta|^2$. This quantity is greater than unity, that is, amplification occurs for those modes that satisfy the superradiance condition $\sigma_\alpha < 0$.

(d) *Loss of energy and angular momentum by a black hole in the process of quantum radiation.* If Φ_{out} is the image of the field ϕ on $\mathscr{I}^+$, the rates at which the system dissipates energy, $\mathrm{d}E/\mathrm{d}u$, and angular momentum, $\mathrm{d}J/\mathrm{d}u$ through the emission of this field are given by the expressions [cf. (5.1.18)]

$$-\frac{\mathrm{d}E}{\mathrm{d}u} = \int \mathrm{d}\Omega \frac{\partial}{\partial u}\Phi_{\text{out}} \frac{\partial}{\partial u}\Phi_{\text{out}}, \tag{9.5.20}$$

$$-\frac{\mathrm{d}J}{\mathrm{d}u} = \int \mathrm{d}\Omega \frac{\partial}{\partial u}\Phi_{\text{out}} \frac{\partial}{\partial \phi}\Phi_{\text{out}}, \tag{9.5.21}$$

where the integration is carried out over the sphere S^2, u = const on $\mathscr{I}^+$, and $\mathrm{d}\Omega = \sin\theta\, \mathrm{d}\theta \mathrm{d}\phi$ is an element of surface on S^2.

In the quantum case, Φ_{out} must be replaced with the operator $\hat{\Phi}_{\text{out}}$, the expression must be symmetrized, and (9.5.20) and (9.5.21) must be averaged over some state corresponding to a certain choice of initial conditions. Special regularization rules must also be fixed because the obtained expression involves field operators at the coinciding points. In the case of interest here, these rules are extremely simple; they reduce to taking the normal form of an expression with respect to the operators $\hat{\beta}^*_\alpha$ and $\hat{\beta}^*_\alpha$ instead of the expression itself. The divergence discussed above is caused by vacuum zero oscillations that result in such divergences, even in the absence of external fields. The problem of interest here is to analyze the effect of the external field on the state of a quantum system. Therefore, what is important now is not the energy and momentum fluxes due to vacuum fluctuations in the flat space but only the changes in these fluxes (measurable by instruments) caused by the formation of the black hole; hence, we have to subtract the expectation values over the out-vacuum state from the respective quantities of interest. It is not difficult to verify that this procedure is equivalent to the transition, described above, to the normal form.

For the sake of simplification, we consider only the case of systems that are in the vacuum state prior to the formation of the black hole. Let

$$G(\xi, \xi') = \langle 0;\ \text{in}|\ :\hat{\Phi}_{\text{out}}(u, \theta, \phi)\hat{\Phi}_{\text{out}}(u', \theta', \phi'):\ |0;\ \text{in}\rangle \tag{9.5.22}$$

$[\xi = (u,\theta,\phi), \xi = (u',\theta';\phi')]$; then

$$-\frac{\mathrm{d}E}{\mathrm{d}u} = \int \mathrm{d}\Omega \left[\frac{\partial}{\partial u}\frac{\partial}{\partial u'} G(u, \theta, \phi, u', \theta, \phi)\right]_{u'=u}, \tag{9.5.23}$$

$$-\frac{dJ}{du} = \int d\Omega \left[\frac{\partial}{\partial u} \frac{\partial}{\partial \phi'} G(u, \theta, \phi; u', \theta, \phi') \right]_{\substack{u'=u \\ \phi'=\phi}} . \tag{9.5.24}$$

Using expansion (9.5.16) of the operator $\hat{\Phi}_{\rm out}$ and relation (9.4.59) for the calculation of $G(\xi, \xi')$, we obtain

$$G(\xi, \xi') = \left\{ \sum_{\alpha,\alpha'} \left[U_\alpha(\xi) \frac{\partial}{\partial \psi_\alpha} + \bar{U}_\alpha(\xi) \frac{\partial}{\partial \bar{\psi}_\alpha} \right] \times \right.$$

$$\left. \times \left[U_{\alpha'}(\xi) \frac{\partial}{\partial \psi_{\alpha'}} + \bar{U}_{\alpha'}(\xi) \frac{\partial}{\partial \bar{\psi}_{\alpha'}} \right] Z[\psi, 0; 0] \right\}_{\psi=0} = \tag{9.5.25}$$

$$= \sum_\alpha \langle \hat{n}_\alpha \rangle_0 \, [\bar{U}_\alpha(\xi) U_\alpha(\xi') + \bar{U}_\alpha(\xi') U_\alpha(\xi)],$$

where $\langle \hat{n}_\alpha \rangle_0$ is the mean number of particles in the mode α that the black hole emits [it is given by (9.5.1)]. Note that any other complete system of functions on $\mathcal{J}^+$ can be used instead of U_α because the summation in (9.5.25) is carried out over a complete system of functions. Thus, it is convenient to rewrite (9.5.25) in the form

$$G(\xi, \xi') = \int_0^\infty d\omega \sum_{l,m} \langle \hat{n}_{\omega l m} \rangle_0 \, [\bar{U}_{\omega l m}(\xi) U_{\omega l m}(\xi') + \bar{U}_{\omega l m}(\xi') U_{\omega l m}(\xi)], \tag{9.5.26}$$

where

$$U_{\omega l m}(\xi) = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega u} Y_l^m(\theta, \phi). \tag{9.5.27}$$

Substituting this expansion of G into (9.5.23) and (9.5.24), carrying out the differentiation, and then integrating in angular variables, using normalization conditions (9.4.4), we finally obtain

$$-\frac{dE}{du} = \frac{1}{2\pi} \int_0^\infty d\omega \sum_{l,m} \frac{\sigma_{\omega l m}\, \omega |T_{\omega l m}|^2}{\exp(\tilde{\omega}/\theta) - 1}, \tag{9.5.28}$$

$$-\frac{dJ}{du} = -\frac{1}{2\pi} \int_0^\infty d\omega \sum_{l,m} \frac{\sigma_{\omega l m}\, m |T_{\omega l m}|^2}{\exp(\tilde{\omega}/\theta) - 1}. \tag{9.5.29}$$

If the black-hole mass is large so that the temperature is very low, then

$$\frac{\sigma_{\omega l m}}{\exp(\tilde{\omega}/\theta) - 1} \approx \frac{1}{2}\,(1 - \sigma_{\omega l m}). \tag{9.5.30}$$

In this limit, only the modes satisfying the superradiance condition contribute to the radiation, and we have

$$-\frac{\mathrm{d}E}{\mathrm{d}u} = \frac{1}{2\pi}\sum_{l,m}\int_0^{m\Omega^H} \mathrm{d}\omega\, \omega |T_{\omega l m}|^2, \tag{9.5.31}$$

$$-\frac{\mathrm{d}J}{\mathrm{d}u} = \frac{1}{2\pi}\sum_{l,m}\int_0^{m\Omega^H} \mathrm{d}\omega m |T_{\omega l m}|^2. \tag{9.5.32}$$

(e) *Entropy of black-hole radiation.* We define the entropy S of a system described by a density matrix $\hat{\rho}$ by the relation

$$S = \mathrm{Sp}(\hat{\rho}\ln\hat{\rho}). \tag{9.5.33}$$

It is readily shown that if $\hat{\rho}$ is written in the form

$$\hat{\rho} = \prod_{\alpha}\hat{\rho}_{\alpha}, \tag{9.5.34}$$

where $\hat{\rho}_{\alpha}$ is an operator and depends only on the creation operators $\hat{\beta}^*_{\alpha}$ and annihilation operators $\hat{\beta}_{\alpha}$ in the mode α, then S can be represented as a sum,

$$S = \sum_{\alpha} S_{\alpha}, \tag{9.5.35}$$

where

$$S_{\alpha} = \mathrm{Sp}_{\alpha}(\hat{\rho}_{\alpha}\ln\hat{\rho}_{\alpha}). \tag{9.5.36}$$

Here Sp_{α} denotes the operation of trace calculation in the space generated by the action of $\hat{\beta}^*_{\alpha}$ on the vacuum.

We have already mentioned that the reflection coefficient R_{jnlm} of a wave packet v_{jnlm} is independent of n for sufficiently large $n \geq N$. Hence, the operator ρ_{α} defined by relation (9.4.35) is also independent of n and, therefore, expression (9.5.35) for the radiation of a stationary black hole is divergent. In view of this, it is convenient, instead of using the total radiated entropy S, to operate with the rate of entropy increase due to radiation in the external space, $\mathrm{d}S/\mathrm{d}u$. Correspondingly we note that the wave packets u_{jnlm} with fixed value of n reach $\mathscr{I}^+$ in the interval of retarded time from $2\pi(n - \frac{1}{2})E$ to $2\pi(n + \frac{1}{2})/E$. We thus define ($\alpha = jnlm$)

$$\frac{\mathrm{d}S}{\mathrm{d}u} = \frac{E}{2\pi}\sum_{jlm}\mathrm{Sp}_{\alpha}(\rho_{\alpha}\ln\rho_{\alpha}). \tag{9.5.37}$$

Now we note that the quantities in this expression are smooth functions of frequency ω and vary slightly when ω changes in the interval form jE to $(j+1)E$. Hence, the summation over j can be replaced with integration in frequency ω:

$$E\sum_j(\ldots) = \int_0^\infty d\omega(\ldots). \tag{9.5.38}$$

As a result, we use expression (9.4.35) for $\hat{\rho}_\alpha$ and obtain

$$\frac{dS}{du} = \frac{1}{2\pi}\int_0^\infty d\omega \sum_{l,m}\left[\frac{\Gamma_\alpha}{z_\alpha - \varepsilon}\ln\left(\frac{z_\alpha - \varepsilon}{\Gamma_\alpha} + \varepsilon\right) + \varepsilon \ln\left(1 + \frac{\varepsilon\Gamma_\alpha}{z_\alpha - \varepsilon}\right)\right], \tag{9.5.39a}$$

where

$$\Gamma_\alpha = \sigma_\alpha |T_\alpha|^2, \quad z_\alpha = \exp(\tilde{\omega}_\alpha/\theta), \quad \varepsilon = 1.$$

This expression is also valid for Fermi fields if we set $\varepsilon = -1$. A similar expression holds for fields of different spins and for massive fields. The summation then extends to all quantum numbers that enumerate the state, and integration begins with μ if the mass of the field is μ. The contribution of the neutrino ($s = \frac{1}{2}$), photon ($s = 1$), and gravitational ($s = 2$) fields into the entropy of the radiation of nonrotating black hole can be written in the form [Page (1983)]

$$\frac{dS}{du} = 10^{-3} M^{-1}\left(1.685h\left(\frac{1}{2}\right) + 0.634h(1) + 0.065h(2)\right), \tag{9.5.39b}$$

where $h(s)$ is the number of polarizations of the spin-s field.

(f) *Probability distribution.* The probability $P(k_\alpha \mid l_\alpha)$ of finding k_α scalar massless particles in the state u_α in the black-hole radiation, provided the flux incoming to the hole contains l_α particles in the state v_α, is given by the general expression (9.4.60). It can be shown that this expression can be transformed as follows [Bekenstein and Meisels (1977), Papangaden and Wald (1977)]:

$$P(k_\alpha|l_\alpha) = (l_\alpha)!(k_\alpha)!\frac{w_\alpha^{2k_\alpha}(1 - w_\alpha^2)(1 - |R_\alpha|^2)^{k_\alpha + l_\alpha}}{(1 - w_\alpha^2|R_\alpha|^2)^{k_\alpha + l_\alpha + 1}} \times$$

$$\times \sum_{n=0}^{\min(k_\alpha, l_\alpha)}\left\{\frac{1}{(l_\alpha - n)!(k_\alpha - n)!(n!)^2}\left[\frac{(1 - w_\alpha^2)^2|R_\alpha|^2}{(1 - |R_\alpha|^2)^2 w_\alpha^2}\right]^n\right\}. \tag{9.5.40}$$

[Analogous expressions for the case of massless fields with higher spin $s = 1/2$, 1, and 2 can be found in Frolov and Bolashenko (1989).]

It is easy to show that

$$e^{-(\tilde{\omega}_\alpha/\theta)l_\alpha} P(k_\alpha|l_\alpha) = e^{-(\tilde{\omega}_\alpha/\theta)k_\alpha} P(l_\alpha|k_\alpha). \tag{9.5.41}$$

We will prove below that this condition ensures detailed equilibrium of the black hole and a cavity rotating at an angular velocity Ω^H and filled with black-body radiation at temperature θ.

(g) *The black hole in a 'thermal bath'*. If there is no matter outside the black hole, the Hawking radiation is the only process that changes the state of a stationary black hole. If there is a matter or radiation outside the black hole, the Hawking evaporation is accompanied by the process of accretion of this matter and radiation on the black hole. It appears that certain matching of the parameters of matter distribution around the black hole to black hole parameters produces an equilibrium situation in which the loss of particles through accretion in each mode is exactly compensated for by the black-hole radiation in this mode. In the simplest case of negligible interaction between different species of particles, these equilibrium conditions must obviously hold for each species of particles individually. Now we will discuss the conditions of equilibrium of a black hole and a gas of massless scalar particles.

Assume that the density matrix $\hat{\rho}_{\text{in}}$ that describes the state of such particles outside a stationary black hole (on $\mathscr{I}^-$) is of the form

$$\hat{\rho}_{\text{in}} = \prod_{\substack{\alpha \\ n \geq N}} \hat{\rho}_{\text{in},\alpha},$$

$$\hat{\rho}_{\text{in},\alpha} = \rho_{0,\alpha} : \exp[-(1+\mu_\alpha)\hat{a}^*_{\text{in},\alpha}\hat{a}_{\text{in},\alpha}] : . \tag{9.5.42}$$

It can be verified [Frolov (1986)] that under the outlined choice of initial condition (9.5.42) the density matrix $\hat{\rho}_{\text{out}}$ that describes the black-hole radiation on $\mathscr{I}^+$ is given by the following expression:

$$\hat{\rho}_{\text{out}} = \prod_{\substack{\alpha \\ n \geq N}} \hat{\rho}_{\text{out},\alpha},$$

$$\hat{\rho}_{\text{out},\alpha} = \rho_{0,\alpha} D_\alpha : \exp[-(1+\mu_\alpha) D_\alpha \hat{\beta}^*_\alpha \hat{\beta}_\alpha] :, \tag{9.5.43}$$

$$D_\alpha = \frac{1 - w_\alpha^2}{1 + \mu_\alpha - (\mu_\alpha + w_\alpha^2)|R_\alpha|^2}.$$

The equilibrium condition, signifying that the distribution of outgoing particles is identical to that of incoming ones, is equivalent to the condition $D_\alpha = 1$ which is satisfied if and only if

$$\mu_\alpha = -w_\alpha^2 = -\exp(-\tilde{\omega}_\alpha/\theta). \tag{9.5.44}$$

Assume that the black hole we consider is enclosed in a stationary axially symmetric, perfectly reflecting ('mirror') surface. Particles reflected by this surface conserve their angular momentum and energy. The action of this surface on wave packets transforms the mode v_α into the mode $-u_\alpha$. If condition (9.5.44) is satisfied, the radiation in the cavity is in equilibrium with the black hole and the density matrix corresponding to this equilibrium state is

$$\hat{\rho}_\theta = \prod_\alpha \hat{\rho}_\alpha,$$

$$\hat{\rho}_\alpha = \rho^0_\alpha \exp\left(-\frac{\tilde{\omega}_\alpha}{\theta} \hat{\beta}^*_\alpha \hat{\beta}_\alpha \right), \tag{9.5.45}$$

where ρ^0_α is normalization constant, and $\hat{\beta}^*_\alpha$ and $\hat{\beta}_\alpha$ are the operators of creation and annihilation in the mode α.

This result can be described somewhat differently. If Sp $\hat{\rho}_\alpha=1$, the probability of finding k_α quanta in the mode α for distribution (9.5.45) is

$$P(k_\alpha) = \frac{e^{-(\tilde{\omega}_\alpha/\theta)k_\alpha}}{1 - e^{-\tilde{\omega}_\alpha/\theta}}. \tag{9.5.46}$$

This relation implies, after equality (9.5.41) is taken into account for the conditional probability $P(l_\alpha|k_\alpha)$, that the probability $P(l_\alpha|k_\alpha)P(k_\alpha)$ of finding l_α outgoing quanta and k_α incoming quanta in the mode α having a given distribution, is equal to the probability $P(k_\alpha|l_\alpha)P(l_\alpha)$ of finding k_α outgoing and l_α incoming quanta, that is, the detailed balance is satisfied, ensuring the thermodynamic equilibrium between the cavity and the black hole in the given mode.

We must emphasize that the density matrix (9.5.43) is normalizable and actually describes the real physical state only if $\mu_\alpha > -1$. Equilibrium condition (9.5.44) for superradiative modes $\tilde{\omega}_\alpha \le 0$ contradicts the condition of normalizability of the density matrix $\hat{\rho}_{\text{in}}$. The result admits the following interpretation. Consider a rotating black hole enclosed in a mirror-walled cavity. Let an arbitrary (normalizable) distribution of particles exist in the chosen superradiative mode α at some moment of time. Then the scattering of this mode by the black hole increases the number of quanta in it. After reflection by the enclosing mirror, these quanta are again scattered on the black hole and their number is again increase. In other words, the system consisting of a black hole and a shell surrounding it acts as a generator for superradiative modes, so that no equilibrium stationary distribution is possible for such modes.

The conclusion obtained above does not imply that in the general case a rotating black hole cannot be in equilibrium with the radiation gas inside the cavity. It isonly necessary that the size of this cavity is not too large $[r \lesssim (\Omega^H)^{-1}]$, that is, that the system has no superradiative modes.

The following arguments support this conclusion. Note that the wave modes $\phi_{\omega lm} = \exp(-i\omega t + im\phi)f_{\omega lm}(r, \theta)$ are eigenmodes for the operator $\eta^\mu \partial_\mu$, where

$$\eta = \xi_t + \Omega^H \xi_\phi , \qquad (9.5.47)$$

namely,

$$\eta^\mu \partial_\mu \phi_{\omega lm} = -i\tilde{\omega}\phi_{\omega lm}. \qquad (9.5.48)$$

Assume that mirror surface enclosing the black hole coincide with the surface of $\eta\cdot\eta$ = const and that the angular velocity of this surface is Ω^H. An observer at rest on this surface has the velocity

$$u^\mu = \eta^\mu/U, \; U^2 \equiv -\eta\cdot\eta. \qquad (9.5.49)$$

In its reference frame, the mode $\phi_{\omega lm}$ has the frequency $\omega' = \tilde{\omega}/U$. Note that now the equilibrium density matrix (9.5.45) can be written in the form

$$\hat{\rho}_\alpha = \rho_{0,\alpha} \exp\left(-\frac{\omega'_\alpha}{T} \hat{\beta}^*_\alpha \hat{\beta}_\alpha\right), \qquad (9.5.50)$$

where

$$T = \kappa/2\pi U. \qquad (9.5.51)$$

This means that a rotating *absorbing* shell enclosing the black hole does not violate the equilibrium if its temperature is T. If the shell surface does not coincide with the $\eta\cdot\eta$ = const surface, the temperature of this shell at equilibrium must be given by (9.5.51) in which

$$U^2 = -\eta\cdot\eta = \Sigma^{-1}\{\Delta(1 - a\Omega^H \sin^2\theta)^2 - \sin^2\theta[a - \Omega^H(r^2 + a^2)]^2\}. \qquad (9.5.52)$$

Let us refer to the surface outside the hole on which the condition

$$\eta\cdot\eta = 0 \qquad (9.5.53)$$

is satisfied as the 'null cylinder'. The solution of Equation (9.5.53) for the Kerr metric in the Boyer–Lindquist coordinates is of the form

$$|\sin\theta| = \frac{\sqrt{X^2 + 4a\Omega^H} - X}{2a\Omega^H}, \quad X^2 = \frac{[\Omega^H(r^2 + a^2) - a]^2}{\Delta}, \quad r > r_+. \qquad (9.5.54)$$

The real function T, given by relations (9.5.51)–(9.5.52) is defined and bounded in the domain between the event horizon and the 'null cylinder' surface. Correspondingly, the equilibrium state described by density matrix (9.5.45) is possible only if the shell enclosing the black hole also lies in this domain [so that $r_0 \lesssim (\Omega^H)^{-1}$].

For a nonrotating black hole, its equilibrium with the gas of radiation is possible regardless of the size of cavity.[9] The equilibrium requires that the temperature of radiation far from the black hole be equal to its Hawking temperature.

The conclusion on the possibility of thermal equilibrium between a black hole and a radiation gas, provided they have equal temperatures and angular velocities,[10] is of a general nature. Gibbons and Perry (1978) demonstrated that this conclusion can be extended to the case of interacting particles.

(h) *Radiation of a charged rotating black hole. Dependence of radiation on the mass, charge, and spin of particles.* In the general case of a black hole also having, in addition to mass M and angular momentum J, an electric charge Q, the expression for the mean number of particles of mass μ, electric charge $q|e|(q=\pm 1)$, and spin s that the black hole creates, can be written as follows [Hawking (1975)]:

$$\langle n_J \rangle = \frac{\Gamma_J}{\exp[(2\pi\tilde{\omega}_J)/\kappa] - (-1)^{2s}}. \tag{9.5.55}$$

Here the collective subscript J denotes the complete set of quantum numbers that must be specified to describe a mode. The set includes the subscript j enumerating particle species and also carrying the information on particle spin s, frequency or energy ω, spheroidal quantum number l, azimuthal quantum number m, polarization or spirality p, and the sign of the particle charge q. The quantity $\tilde{\omega}_J$ in (9.5.55) is

$$\tilde{\omega}_J = \omega_J - m_J\Omega^H - q_J|e|\Phi^H, \tag{9.5.56}$$

where Ω^H and Φ^H are the angular velocity and electric potential of the black hole, and $\Gamma_J = 1-|R_J|^2$, where R_J is the reflection coefficient of the incident wave J. The coefficient Γ_J becomes negative for superradiance modes of boson fields. In fermion fields, the Pauli principle implies that the mean number of particles scattered in a given mode cannot exceed unity, so that always $|R_J|^2 \leq 1$. The coefficient Γ_J is positive and, hence, the expression in the right-hand side of (9.5.55) is always positive. Superradiance is not possible for fermions [Martellini and Treves (1977), Iyer and Kumar (1979), Chandrasekhar (1979a, b)].

We denote by Σ_J the summation over all discrete quantum numbers and the integration over continuous ones in J,

$$\sum_J = \sum_{j,l,m,p,q} \frac{1}{2\pi}\int_\mu^\infty d\omega, \tag{9.5.57}$$

and then obtain the following formulas for the rate of change, due to quantum radiation, of mass, angular momentum, and charge of the black hole:

$$-\frac{dM}{dt} = \sum_J \omega_J\langle n_J\rangle\,, \quad -\frac{dJ}{dt} = \sum_J m_J\langle n_J\rangle\,, \quad -\frac{dQ}{dt} = e\sum_J q_J\langle n_J\rangle\,. \tag{9.5.58}$$

The calculation of the contributions of individual particle species into quantum radiation of a black hole reduces to the determination of the respective reflection coefficients for wave functions describing these particles. A considerable number of papers have been devoted to analyzing reflection coefficients for various fields in the Kerr-Newman metric or its particular cases and to developing the methods of

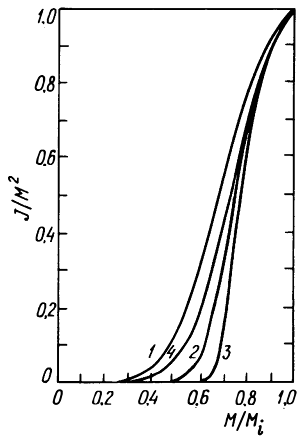


Fig. 76. The evolution of the rotation parameter J/M^2 of a black hole in the course of evaporation (M_i is the initial and M is the current mass of the black hole). The figure shows the behavior of J/M^2 as a function of M/M_i in the case of only a single neutrino field (1), only the photon field (2), or only the graviton field (3), and in the real situation (4) (four species of neutrino, one photon, and one graviton species).

approximate description of coefficients. For a complete representation of the relevant mathematical aspects and a review of the results obtained, we refer the reader to Chandrasekhar's monograph (1983). In this section we dwell only on those results of this analysis that are of special physical interest.

(1) Loss of electric charge by a black hole was analyzed by Markov and Frolov (1970), Zaumen (1974), Carter (1974), Gibbons (1975), Nakamura and Sato (1976), Damour and Ruffini (1975), Page (1977), Ruffini (1979), Novikov and Starobinsky (1980c). The main result of this analysis is as follows. Lone black holes of mass $M \leq Ge^2/m \approx 10^{15}$g ($m$ is the electron mass) shed their electric charge almost

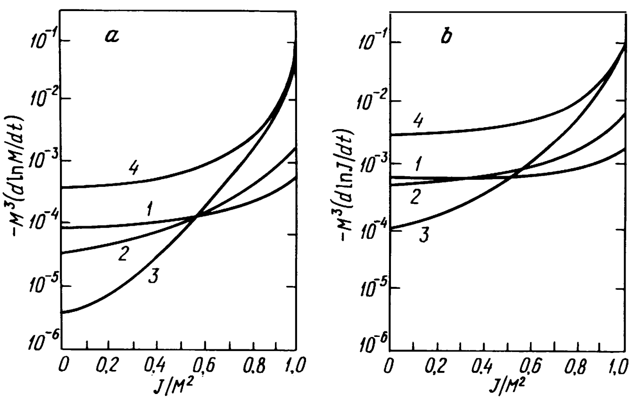


Fig. 77. Energy (a) and angular momentum (b) radiative power of a black hole as a function of rotation parameter J/M^2. The contributions of a single neutrino species (1), photons (2), and gravitons (3) are plotted separately, as well as the total radiative power for massless particles in a real situation (4) (four species of neutrinos, one photon and one graviton).

completely and very rapidly. The time necessary for radiating away the electric charge of black holes with mass $M < \sqrt{G}\, e m_{Pl}^2/m^2 \approx 10^5 M_\odot$ is much shorter than the characteristic time of evaporation of a black hole (see below in Section 12.3 and Figure 85). Therefore, a black hole can be treated as electrically neutral during nearly entire period of its evaporation.

(2) Dimensions-based arguments show that, in the general case, a black hole can dissipate angular momentum over a time comparable with a black hole evaporation time. Carter (1974) hypothesized that in the course of evaporation, the ratio of angular momentum to squared mass of the black hole tends to a nonzero limiting value. However, Page's numerical results (1976b) refuted this conjecture. If the contribution of really existing massless particles (neutrinos, photons, and gravitons) is taken into account, the angular momentum is found to be dissipated several times faster than the black hole mass (Figure 76). It is also found that the smaller the spin of massless particles, the greater their contribution to the radiation of mass and angular momentum of slowly rotating black holes ($J/M^2 \lesssim 0.6$ and the number of polarization states is assumed to equal 2 for all particles). The situation is opposite for rapidly rotating black holes ($J/M^2 > 0.6$), the contribution being the greater, the higher the particle spin (Figure 77). This effect is in agreement with that of Starobinsky and Churilov (1973) on the spin dependence of superradiance.

Note also that a rotating black hole emits neutrinos asymmetrically, namely, neutrinos are more abundant in the radiation along the rotation direction, while antineutrinos are more abundant in the opposite direction [Unruh (1973), Vilenkin (1979b), Leahy and Unruh (1979)]. This effect has an analogue in the radiation of photons and gravitons by a rotating black hole. Namely, there is an asymmetry in the emission of left- and right-hand-polarized quanta in a given direction. As a result, the electromagnetic and gravitational radiation of a black hole are polarized

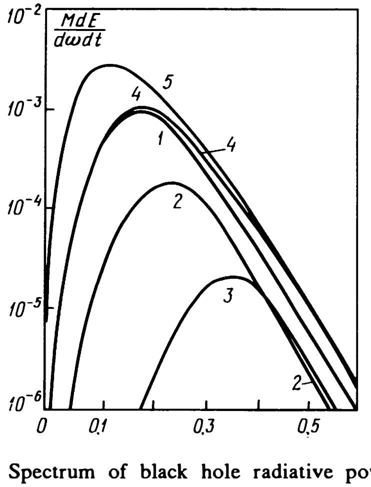


Fig. 78. Spectrum of black hole radiative power. The curves plot the contributions of four neutrino species (1), photons (2), and gravitons (3), and the aggregate spectrum (4). For the sake of comparison, the blackbody radiation spectrum of these particles with cross-section $27\pi M^2$ is given (curve 5).

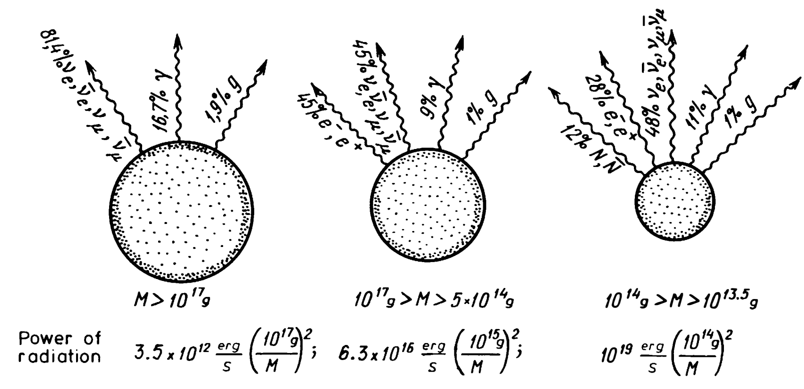


Fig. 79. Quantum decay of nonrotating black hole. The fractions of gravitons (g), photons (γ), neutrinos (ν) and other elementary particles are given in percent of the total number of particles emitted by black holes of different masses.

[Dolgov *et. al.* (1988b), Bolashenko and Frolov (1987, 1989)]. Dolgov *et al.* (1988a) pointed out that these effects are directly related to the so-called chiral anomalies in the gravitational field [Delbourgo and Salam (1972), Eguchi and Freund (1976)]. On other effects in black holes due to chiral anomalies, see Gal'tsov (1986), Gal'tsov *et al.* (1988).

(3) A nonrotating black hole emits almost as a body heated to the temperature

$$\theta = \frac{\hbar\kappa}{2\pi c} = \frac{\hbar c^3}{8\pi GkM} \approx 10^{26}\mathrm{K}(M/1\mathrm{g})^{-1}.$$

The deviation from thermal radiation occurs because the coefficient Γ_J is frequency-dependent. At high frequencies, the effective cross-section of the black hole is $27\pi G^2M^2/c^4$ for all particles. The cross-section decreases at low frequencies, and is found to be strongly spin-dependent:

$$\Gamma_J \propto \omega^{2s+1}, \tag{9.5.59}$$

where $s \geq 0$ is the spin of the field. As spin increases, the contribution of particles to the total radiation of a nonrotating black hole diminishes [Page (1976a)] (Figure 78). Black holes of mass $M > 10^{17}$ can emit only massless particles: neutrinos (ν), photons (γ), and gravitons (g). Black holes of mass $5 \times 10^{14}g \leq M \leq 10^{17}$ can also emit electrons and positrons. Black holes of smaller mass can emit heavier elementary particles as well. The distribution of black-hole products in different mass intervals in shown in Figure 79.

The following formula describes the rate of mass loss by an evaporating black hole:

$$-\frac{\mathrm{d}M}{\mathrm{d}t} \approx 4 \times 10^{-5}\left(\frac{m_{\mathrm{Pl}}}{M}\right)^2 \frac{m_{\mathrm{Pl}}}{t_{\mathrm{Pl}}} f = 7.7 \times 10^{24}\left(\frac{M}{1_g}\right)^{-2} f\,\frac{g}{s}\,. \qquad (9.5.60$$

The function $f = f(M)$ takes into account the dependence of dM/dt on the number of states and species of particles that contribute to the radiation of a black hole of mass M. The contributions of massless particles [neutrinos $s = 1/2$), photons ($s = 1$), and gravitons ($s = 2$)] are given by the formula [Page (1976a, b)]

$$f = 1.023\, h(\frac{1}{2}) + 0.420\, h(1) + 0.048\, h(2), \tag{9.5.61}$$

where $h(s)$ is the number of distinct polarizations of spin-s particles.

The lifetime of a black hole with respect to its quantum evaporation is

$$t \approx 2.7 \times 10^{-26} s \int_0^M \frac{M^2\, dM}{f(M)} \lesssim 9 \times 10^{-27} s(M/1_g)^3. \tag{9.5.62}$$

This time does not exceed the age of the Universe for black holes of mass $M < 5 \times 10^{14}$g. Such black holes could be born as primordial holes at early stages in the evolution of the Universe (see Section 13.1). As for the possibility of observing quantum explosions of such small black holes and the products of their decay, see Carr's review (1983) and the references therein.

(4) External factors could be used to affect the radiation of a black hole and, to some extent, control it. Thus, the radiative power and other characteristics of black-hole radiation change when an external field is switched on. For example, when a black hole is 'inserted' into an external gravitational potential, the intensity of its radiation at infinity diminishes in complete accordance with the reduction in temperature as measured by a distant observer [Geroch and Hartle (1982), Zhuk and Frolov (1981)]. On the effect of magnetic field on particle creation in black holes, see Gibbons (1976), Gal'tsov (198), Sokolov *et al.* (1984). An interesting example of the effect of external factor on the radiative power of a black hole was analyzed by Unruh and Wald (1982) (see also Section 11.3).

Chapter 10

Vacuum Polarization near Black Holes

10.1. Quasiclassical Approximation. Renormalized Energy-Momentum Tensor

Quantum radiation of a lone black hole reduces its mass and, hence, its surface area. This 'violation' of Hawking's theorem is explained by making the inevitable conclusion that the particle flux to infinity that carries away positive energy is accompanied by a flux of negative energy across the horizon into the black hole. This would be impossible in classical theory, with natural physical assumptions satisfied (energy dominance conditions). In quantum theory, negative energy density and (or) negative pressure may arise in some region of space because an external field applied to vacuum may both increase and reduce the local energy density. It is this phenomenon, due to vacuum polarization in a strong gravitational field, that is to be expected in the neighborhood of black holes.

The process of evaporation of a black hole of a mass much greater than the Planck mass, can be described using the *quasi-classical approximation.* Assuming the fluctuation of the gravitational field to be small, we describe it in terms of the classical metric

$$g_{\mu\nu} = \langle g_{\mu\nu} \rangle \tag{10.1.1}$$

satisfying modified Einstein equations

$$G_{\mu\nu} = 8\pi \langle T_{\mu\nu} \rangle, \tag{10.1.2}$$

where the right-hand sides include the expectation value of the energy-momentum tensor of the relevant quantized fields in the chosen state. If the characteristic curvature radius in a region of spacetime, L, is much greater than the Planck length $l_{Pl} = \sqrt{(\hbar G/c^3)}$ the calculation of $\langle T_{\mu\nu} \rangle$ can make use of the expansion in a small parameter $\varepsilon = (l_{Pl}/L)^2$ and retain only the terms up to first order in ε (quasiclassical approach). The first terms of order ε^0 coincide with the expression for the energy-momentum tensor of the classical field, while the term of order ε^1, containing a factor $\hbar$, represents the main contribution of quantum effects (in the chosen approximation of $\varepsilon \ll 1$). This contributions describes the change in energy-momentum density due to the effect of the gravitational field on the state of virtual

vacuum pairs. Other terms of higher order in ε describe additional contributions due to the additional interaction between the particles of a virtual pair as a result of emission and subsequent absorption of field quanta by these particles.[1] In the approximation linear in ε ('one-loop' approximation), virtual pairs of various fields can be treated as noninteracting. Therefore, the contribution of all fields to $\langle T_{\mu\nu}\rangle$ are additive in this approximation and thus can be studied independently.

The main problem in analyzing $\langle T_{\mu\nu}\rangle$ stems for its divergence. To put it more exactly, any calculation involving the need to find the expectation of a quantity containing a product of two or more operators of the field in the same points ($T_{\mu\nu}$ has precisely this form) results in infinities. Such divergences, arising even in flat spacetime, are generated by vacuum zero fluctuations. The methods of extracting a finite, physically meaningful part of $\langle T_{\mu\nu}\rangle$, known as renormalization procedures, were widely discussed in the literature in connection with the development of general field theory in curved spacetime and with its specific applications to cosmology and black-hole physics. A detailed discussion of these problems can be found in DeWitt (1965, 1975), Grib *et al.* (1980), Birrel and Davies (1982), and Christensen (1976, 1978). For this reason, we only briefly outline the renormalization procedure for $\langle T_{\mu\nu}\rangle$ and go into some detail of those aspects of the vacuum polarization effects that are related to black-hole specifics (e.g., the choice of the vacuum state). We also give the main results of calculating $\langle T_{\mu\nu}\rangle^{\text{ren}}$.

A considerable number of renormalization methods have been suggested (dimensional regularization, ζ-function method, Pauli–Villars regularization, n-wave regularization, adiabatic regularization, and the splitting point approach). It is important, however, that the final results are essentially independent on the choice of renormalization method. The point is that, as was shown by Wald (1977, 1978a, b), all renormalization methods for $\langle T_{\mu\nu}\rangle$, that (1) preserve the general covariance ($\nabla_\mu\langle T^{\mu\nu}\rangle^{\text{ren}} = 0$); (2) satisfy natural causality requirements; (3) preserve the value of $\langle\Psi|T_{\mu\nu}|\Phi\rangle$ for those states $|\Psi\rangle$ and $|\Phi\rangle(\langle\Psi|\Phi\rangle = 0)$ for which this quantity is finite; and (4) agree with standard procedure of normal ordering in flat spacetime, result in expressions for $\langle T_{\mu\nu}\rangle^{\text{ren}}$ that can differ from one another only in a local conserved tensor constructed of the curvature tensor at a given point and its covariant derivatives.

The possible ambiguity in $\langle T_{\mu\nu}\rangle^{\text{ren}}$ in the one-loop approximation must be described by an expression formed by the sum of the terms quadratic in curvature and terms linear in its second derivatives, because massless fields are not characterized by a field-related parameter of length dimension. It is impossible to construct such a symmetric conserved second-rank tensor using only the Weyl tensor. Hence, the above-mentioned ambiguity in the determination of $\langle T_{\mu\nu}\rangle^{\text{ren}}$ is absent in the one-loop approximation for massless fields against the background of a metric satisfying Einstein's vacuum equations ($R_{\mu\nu} = 0$).

In the case of a classical conformally invariant field, $T^\mu_\mu = 0$. An important distinctive feature of $\langle T_{\mu\nu}\rangle^{\text{ren}}$ is that the trace of this quantity does not vanish any more for the conformally invariant field (this effect is known as the *conformal trace anomaly*). The quantity $\langle T^\mu_\mu\rangle^{\text{ren}}$ is independent of the choice of the state over which $T_{\mu\nu}$ is averaged. In the case of the scalar conformally invariant field ($s = 0$), two-

component massless spinor field ($s = 1/2$), and electromagnetic field ($s = 1$), $\langle T^{\mu}_{\mu}\rangle^{\rm ren}$ is written in the form [2]

$$\langle T^{\mu}_{\mu}\rangle^{\rm ren} = a_s\left(C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} + \frac{2}{3}\Box R\right) + b_s(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2), \quad (10.1.3)$$

where

$$a_s = \frac{\tilde{a}_s}{5760\pi^2}, \quad b_s = \frac{\tilde{b}_s}{5760\pi^2},$$

$$\tilde{a}_0 = 3, \quad \tilde{a}_{1/2} = 9, \quad \tilde{a}_1 = 36, \quad (10.1.4)$$

$$\tilde{b}_0 = -1, \quad \tilde{b}_{1/2} = -\frac{11}{2}, \quad \tilde{b}_1 = -62.$$

As a rule, practical calculations of $\langle T_{\mu\nu}\rangle^{\rm ren}$ for the gravitational field of black holes employ the splitting-point approach. The method is as follows. The stress-energy tensor $T_{\mu\nu}$ is a bilinear function of the fields, so that it is formally possible to generalize $T_{\mu\nu}$ to the case of arguments being different for each of the fields (to 'split the points'). In the classical theory, $T_{\mu\nu}(x)$ appears as a limit of the corresponding expression $T_{\mu\nu'}(x, x')$ at $x' = x$. When the expectation value of the operator $\hat{T}_{\mu\nu'}(x, x')$ in a given state $T_{\mu\nu'}(x, x') = \langle\hat{T}_{\mu\nu'}(x, x')\rangle$ is calculated in the quantum theory, a simple transition to limit $x' \to x$ is impossible because divergences appear. For this reason, the quantity $T_{\mu\nu'}(x, x')$ is 'corrected' (renormalized), before taking the limit, by subtracting from it the standard expression $T^{\rm div}_{\mu\nu'}(x, x')$. It is sufficient to calculate $T^{\rm div}_{\mu\nu'}$ once for each of the fields. Christensen (1978) derived the expressions for $T^{\rm div}_{\mu\nu'}$ in the cases of the massless scalar and spinor fields and for the electromagnetic field.

When matrix elements $\langle\Psi|\, T_{\mu\nu}(x)\, |\Phi\rangle^{\rm ren}$ of the energy-momentum tensor of the field ϕ_A are calculated for the specified states $|\Phi\rangle$ and $|\Psi\rangle$, it is convenient to use Green's function

$$G_{AB'}(x, x') = i\,\frac{\langle\Psi|T(\hat{\phi}_A(x)\hat{\phi}_{B'}(x'))|\Phi\rangle}{\langle\Psi|\Phi\rangle}. \quad (10.1.5)$$

The symbol T in (10.1.5) denotes the operation of T-product,

$$T(\hat{\phi}_A(x)\hat{\phi}_{B'}(x')) = \theta(x, x')\hat{\phi}_A(x)\hat{\phi}_{B'}(x') + \theta(x', x)\hat{\phi}_{B'}(x')\hat{\phi}_A(x), \quad (10.1.6)$$

where $\theta(x, x')$ is the step function equal to 1 if x lies in the future relative to x', and equal to 0 otherwise. Using commutation relations (9.2.5), it is easy to show that Green's function (10.1.5) for the field ϕ_A, described by Equation (9.2.2), satisfies the following equation:

$$D^{AC}G_{CB'}(x, x') = -\delta^A_{B'}\,\delta^4(x - x'), \quad (10.1.7)$$

where $\int \delta^4(x - x')\, d^4x = 1$. This Green's function is related to the so-called *Hadamard's function*:[3]

$$G^{(1)}_{AB'}(x, x') = \frac{\langle \Phi | \hat{\phi}_A(x)\hat{\phi}_{B'}(x') + \hat{\phi}_{B'}(x')\hat{\phi}_A(x) | \Psi \rangle}{\langle \Phi | \Psi \rangle} \quad . \tag{10.1.8}$$

The values assumed by the energy-momentum tensor at the split points are

$$T_{\mu\nu'}(x, x') = \frac{1}{2} T^{AB'}_{\mu\nu'}(x, x') G^{(1)}_{AB'}(x, x'), \tag{10.1.9}$$

where $T^{AB'}_{\mu\nu'}(x, x')$ is the differential operator with respect to variables x and x', such that the quantity $T^{AB'}_{\mu\nu'}(x, x')\phi_A(x)\phi_{B'}(x)$ coincides with $T_{\mu\nu}(x)$. The explicit form of the operators $T^{AB'}_{\mu\nu'}$ for different-spin fields is given in Christensen's paper (1978).

In addition to $\langle T_{\mu\nu}\rangle^{\text{ren}}$, one often considers quantities of type $\langle \phi_A^2\rangle^{\text{ren}}$ that describe the fluctuations of the field ϕ_A. In the case of a scalar field ϕ, the quantity $\langle \phi^2\rangle$ was analyzed in the field of a black hole in connection with the possibility of phase transition in its neighborhood. These transitions consist of the appearance of $\langle \phi(x)\rangle \neq 0$ [Hawking (1981), Fawcett and Whiting (1982), Moss (1984)]. The following simple expression for the renormalized quantity $\langle \phi^2\rangle^{\text{ren}}$ holds in the gravitational field described by Einstein's vacuum equations:

$$\langle \phi^2(x) \rangle^{\text{ren}} = \lim_{x' \to x} \left[-iG(x, x') - \frac{1}{8\pi^2 \sigma(x, x')} \right], \tag{10.1.10}$$

where $\sigma(x, x') = \frac{1}{2}s^2(x, x')$, $s(x, x')$ is the interval of geodesic distance between x and x', and x' tends to x along a space-like direction.

The problem of calculating $\langle T_{\mu\nu}\rangle^{\text{ren}}$ and $\langle \phi^2\rangle^{\text{ren}}$ which carry the information on energy-momentum density (related to the vacuum polarization) and on vacuum fluctuations, thus reduces to performing a set of standard operations over Green's function (10.1.5). As a result, the solution of the problem of vacuum polarization in black holes reduces to solving Equation (10.1.7) in a given metric describing the spacetime of a black hole. The ambiguity in the choice of state used to carry out the averaging in (10.1.5) corresponds to the arbitrariness in the choice of the boundary conditions that unambiguously specify a concrete Green's function.

10.2. Choice of State and Boundary Conditions for Green's Functions

Now we will describe the states and corresponding boundary conditions for Green's functions that are used more frequently than others in considering quantum effects in black holes.[4] For the sake of simplification, we limit the analysis to the case of a massless scalar field.

The situation of obvious interest is that of a black hole arising as a result of collapse and the quantum system being originally in the ground, vacuum state. The corresponding Green's function,

$$G_{\text{in}}(x, x') = i\,\langle 0;\, \text{in}|T(\hat{\phi}(x)\hat{\phi}(x'))|0;\, \text{in}\rangle\,, \tag{10.2.1}$$

being symmetric as are all other Green's functions (10.1.5), is determined unambiguously by the following property: It coincides in the distant past (in the in-region) with the free causal Green's function in Minkowski space. Obviously, the behavior of G_{in} generally depends on details of the collapse forming the black hole. As a result, the determination of G_{in} is a complicated problem, difficult to solve. Recall, however, that after some time, the black hole becomes stationary.[5] The arguments used to prove the universality of the properties of the Hawking radiation at later time lead to the conclusion that after a sufficiently long time after the formation of the black hole, G_{in} is determined exclusively by the black-hole parameters.

The following method is convenient for describing the properties of this 'universal' Green's function. Consider the spacetime of an eternal black hole having the same parameters as the arising stationary black hole. In this spacetime, define a Green's function $G_U(x, x')$ as the solution of the Equation (10.1.7) that at later time coincides with the asymptotic form of $G_{\text{in}}(x, x')$ and is obtained as its analytic continuation in the remaining spacetime. Unruh (1976b) proved that the following boundary conditions define Green's function $G_U(x, x')$ unambiguously in the spacetime of an eternal black hole for the values of arguments x, x' that lie in the exterior region or at the event horizon: This function for a fixed value of x' in region I (Figure 80) is of negative frequency with respect to the affine parameter U on H^- for x on H^-, and of negative frequency in advanced time v on $\mathscr{I}^-$ for x on $\mathscr{I}^-$. The corresponding state $|U\rangle$ for which

$$G_U(x, x') = i\langle U|\, T(\hat{\phi}(x)\hat{\phi}(x'))\, |U\rangle \tag{10.2.2}$$

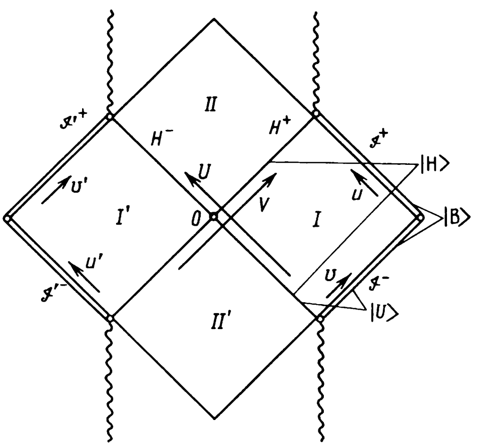


Fig. 80. Penrose diagram for rotating black hole. The straight lines point to surfaces bounding the region *I*, where boundary conditions are fixed for Green's functions corresponding to the Hartle–Hawking ($|H\rangle$), Unruh ($|U\rangle$), and Boulware ($|B\rangle$) vacuum states

was called the *Unruh vacuum.*

Another case of interest is that of a black hole placed in a cavity in equilibrium with the black-body radiation filling the cavity. Since this state is not pure and is described by density matrix $\hat{\rho}_\theta$ (9.5.45), Green's function $G_H(x,x')$ corresponding to it is

$$G_H(x, x') = i\mathrm{Sp}[\hat{\rho}_\theta T(\hat{\phi}(x)\hat{\phi}(x'))]. \qquad (10.2.3)$$

This Green's function can also be extended by analytic continuation to the spacetime of the eternal black hole. As shown by Hartle and Hawking (1976), this function satisfies the following boundary conditions: For a fixed value of x' it is a negative-frequency function with respect to the affine parameter U for x on H^- and a positive-frequency function with respect to the affine parameter V for x on H^+ (see Figure 80). Even though this state is neither pure nor vacuum, it is usually denoted by the symbol $|H\rangle$ and (10.2.3) is then abbreviated to

$$G_H(x, x') = i\langle H|\, T(\hat{\phi}(x)\hat{\phi}(x'))\, |H\rangle\ . \qquad (10.2.4)$$

This state is called the *Hartle–Hawking vacuum.*

As we have mentioned in the preceding chapter, the equilibrium of a rotating black hole in a cavity is possible only if the size of this cavity is sufficiently small. One can accordingly expect that if there is no cavity or if its boundary lies outside the 'null cylinder', Green's function $G_H(x, x')$ is, in a certain sense, pathological. Kay and Wald (1987, 1988) have shown that this is indeed so [see also Frolov and Thorne (1989)]. This is why we assume, when considering below the state $|H\rangle$ in the field of a rotating black hole, that the black hole in question is placed in a cavity comprised within the 'null cylinder'. For a slowly rotating black hole, the boundary of the cavity may lie very far from the black hole, so that its influence is negligible. If the cavity size is small and the boundary effects are important, one needs to specify the boundary conditions at its surface Σ. Usually we assume that

$$\phi|_\Sigma = 0. \qquad (10.2.5)$$

Green's function must also satisfy the appropriate boundary condition:

$$G_H(x, x')|_{x\in\Sigma} = 0. \qquad (10.2.6)$$

If the role of the boundary surface Σ is important, we replace the symbol $|H\rangle$ with $|H, \Sigma\rangle$, and G_H with $G_{H,\Sigma}$.

Hartle and Hawking (1976) demonstrated that Green's function G_H (as well as $G_{H,\Sigma}$) has special analytical properties. In order to describe these properties, note that if a change of variables,

$$t = -i\tau,\ \ a = ib, \qquad (10.2.7)$$

is carried out in the expression for the length element in the Kerr geometry, the resulting metric has the signature $+ + + +$. Moreover, one can show [Hartle and Hawking (1976), Hawking (1981)] that this metric is everywhere regular (including

the surface $r = r_E = M+\sqrt{(M^2 + b^2)}$ corresponding to the analytic continuation of the event horizon surface) provided the coordinate τ is cyclic, with the period $2\pi/\kappa_E(\kappa_E = \kappa|_{a=ib})$. A regular space with this metric is known as the *Euclidean black hole*. The result obtained by Hartle and Hawking (1976) states that Green's function $G_E(x, x')$ produced by the analytic continuation (10.2.7) of the function $G_H(x, x')$,

$$G_H(x, x') = [iG_E(x, x')]_{\substack{\tau = it \\ b = -ia}} , \tag{10.2.8}$$

is a symmetric solution to the equation

$$\Box_E G_E(x, x') = -\frac{\delta(x, x')}{\sqrt{g_E}} \tag{10.2.9}$$

in the space of the Euclidean black hole, falling off at infinity and regular at the surface of the Euclidean horizon. Here

$$\Box_E = \Box|_{\substack{t = -i\tau \\ a = ib}} , \quad g_E = g|_{\substack{t = -i\tau \\ a = ib}} .$$

This result makes it possible to employ the following method for constructing $G_H(x, x')$: first find a Euclidean Green's function G_E and then obtain G_H using analytic continuation (10.2.8). This procedure is very similar to that of Wick's rotation and transition to the Euclidean formulation that is frequently used in quantum field theory in flat spacetime. Quite often, this procedure considerably facilitates the calculation of G_H.

Let us briefly discuss one more choice of state which is useful for describing quantum effects in black holes and was first suggested by Boulware (1975a, b, 1976). This state, denoted by $|B\rangle$, is known as the *Boulware vacuum*. Consider a situation of a nonrotating spherical body of mass M and radius R_0 that is slightly larger than the gravitational radius r_g of this body. Since the corresponding Killing vector field $\xi^\mu_{(t)}\partial_\mu = \partial_t$ in such a static spacetime is everywhere time-like, any particle has positive energy, and particle creation is impossible. The corresponding vacuum state $|B;R_0\rangle$ is stable and the respective Green's function $G_{B;R_0}(x, x')$ satisfies the following boundary conditions: at a fixed value of x', it is a negative-frequency function of v for x on $\mathcal{J}^-$ and a positive-frequency function of u for x on $\mathcal{J}^+$. We define Green's function

$$G_B(x, x') = i\langle B|T(\hat{\varphi}(x)\hat{\varphi}(x'))|B\rangle \tag{10.2.10}$$

as the solution of the equation $G_B(x, x') = -\delta(x, x')/\sqrt{(-g)}$ in the spacetime of an eternal black hole that satisfies the boundary conditions given above.

Green's function G_B for a nonrotating black hole can be treated, in a certain sense, as the limit of $G_{B;R_0}$ as $R_0 \to r_g$ Obviously, no physical realization is possible of a static system whose size is arbitrarily close to the gravitational radius. In this limit, the particles on the surface of the body move at infinitely large acceleration and bringing them to rest requires infinitely large forces. As a result, Green's function G_B that has simple, regular behavior far from the black hole and

corresponds to the absence of quantum radiation on $\mathscr{I}^+$, reveals 'poor' analytical behavior close to the event horizon. Renormalized quantities $\langle B|T_{\mu\nu}|B\rangle$ and $\langle B|\phi^2|B\rangle$ corresponding to G_B diverge on H^+ and H^-.

The states and Green's functions described above were defined only for a scalar massless field. There are no difficulties in extending these definitions to the general case.

10.3. $\langle T^{\mu}_{\nu}\rangle^{\text{ren}}$ and $\langle \phi^2\rangle^{\text{ren}}$ in the Spacetime of a Black Hole

To calculate the renormalized value of the energy-momentum tensor, one needs to know the values that Green's function $G(x,x')$ assumes for close values of x and x'. However, this does not mean that the boundary conditions, imposed on Green's function far from the point of interest for us, do not effect the behavior of $G(x,x')$ in the limit of coinciding points.[6] This is readily verified if one recalls that the equation determines Green's function up to a solution of the homogeneous equation which is fixed unambiguously by the boundary conditions.

If the characteristic curvature of radius L of spacetime in the theory of massive field (with mass m) is much greater than the Compton length $\lambda = \hbar/mc$, one can use the expansion in a small parameter $\varepsilon = (\lambda/L)^2$ and obtain a uniform approximation for Green's function. No such parameter can be used if the field is massless. In the Kerr metric the natural method of analyzing such fields is to expand them in their eigenmodes [Candelas (1980), Candelas *et al.* (1981)], because the wave equations for massless fields permit the separation of variables.

As an example, we will look at the representation for Hadamard's functions of a scalar massless field in the metric of a rotating black hole (the formulas given below are valid for charged black holes as well). As basis solutions of the equation $\Box\phi = 0$ in the spacetime of a black hole, it is convenient to choose the system of solutions $v_{\omega lm}$, $y_{\omega lm}$ that satisfy the following boundary conditions. The functions $v_{\omega lm}$ vanish on H^-; their image on $\mathscr{I}^-$ is $V_{\omega lm}$, described by expression (9.4.5). The image of $y_{\omega lm}$ vanishes on $\mathscr{I}^-$. On H^-, these functions take the values

$$y_{\omega lm}|_{H^-} = \frac{e^{-i\omega u}}{\sqrt{4\pi\tilde{\omega}(r_+^2 + a^2)}} Y_{lm}(\theta, \phi). \tag{10.3.1}$$

If the black hole is surrounded by a mirror shell whose equation is $r = r_0$ ($\phi|_{r_0} = 0$), we choose, as the corresponding basis functions, the solutions $k_{\omega lm}$ which assume on H^- the values coinciding with $y_{\omega lm}|_{H^-}$ [formula (10.3.1)] and vanish on the shell surface: $k_{\omega lm}|_{r_0} = 0$. If the collective subscript ωlm is denoted by J and we use the notation

$$v_J(x, x') = v_J(x)\bar{v}_J(x') + \bar{v}_J(x)v_J(x'),$$

$$y_J(x, x') = y_J(x)\bar{y}_J(x') + \bar{y}_J(x)y_J(x'), \tag{10.3.2}$$

$$k_J(x, x') = k_J(x)\bar{k}_J(x') + \bar{k}_J(x)k_J(x'),$$

then Hadamard's functions $G^{(1)}$ corresponding to various choices of the 'vacuum' states are written in the following form:

$$G_U^{(1)}(x, x') = \sum_{l,m}\left[\int_0^\infty \mathrm{d}\omega\, v_J(x, x') + \int_{\Omega m}^\infty \mathrm{d}\omega \coth\frac{\pi\tilde{\omega}}{\kappa}\, y_J(x, x')\right],$$

$$G_H^{(1)}(x, x')$$

$$= \sum_{l,m}\left[\int_0^\infty \mathrm{d}\omega \coth\frac{\pi\tilde{\omega}}{\kappa} v_J(x, x') + \int_{\Omega m}^\infty \mathrm{d}\omega \coth\frac{\pi\tilde{\omega}}{\kappa} y_J(x, x')\right], \qquad (10.3.3)$$

$$G_B^{(1)}(x, x') = \sum_{l,m}\left[\int_0^\infty \mathrm{d}\omega\, v_J(x, x') + \int_{\Omega m}^\infty \mathrm{d}\omega\, y_J(x, x')\right],$$

$$G_{H;r_0}^{(1)}(x, x') = \sum_{l,m}\int_{\Omega m}^\infty \mathrm{d}\omega \coth\frac{\pi\tilde{\omega}}{\kappa}\, k_J(x, x').$$

Similar representations of Green's functions for the electromagnetic field and gravitational perturbations are given in the paper of Candelas *et al.* (1981). On Green's functions of a scalar field, see also Candelas (1980) and Frolov (1986).

Note that the difference between any two functions (10.3.3) remains finite in the limit of coinciding points, because the divergences removed by renormalization have universal form. Thus, for these finite differences we have

$$\langle U|\phi^2(x)|U\rangle - \langle H|\phi^2(x)|H\rangle = -2\sum_{l,m}\int_0^\infty \frac{\mathrm{d}\omega|v_J(x)|^2}{\mathrm{e}^{-\tilde{\omega}/\theta} - 1}, \qquad (10.3.4)$$

$$\langle B|\phi^2(x)|B\rangle - \langle H|\phi^2(x)|H\rangle$$

$$= -2\sum_{l,m}\left[\int_0^\infty \frac{\mathrm{d}\omega|v_J(x)|^2}{\mathrm{e}^{\tilde{\omega}/\theta} - 1} + \int_{\Omega m}^\infty \frac{\mathrm{d}\omega|y_J(x)|^2}{\mathrm{e}^{\tilde{\omega}/\theta} - 1}\right], \qquad (10.3.5)$$

where $\theta = \kappa/2\pi$ is the black-hole temperature.

Representations of the form (10.3.3) for Green's functions and those analogous to them for higher-spin fields make it possible to analyze the behavior of $\langle\phi^2\rangle^{\text{ren}}$ and $\langle T^\mu_\nu\rangle^{\text{ren}}$ close to $H^\pm$ and $\mathcal{J}^\pm$. One can thus show that close to $H^\pm$ these quantities diverge when averaged over the Boulware vacuum. In the neighborhood of a

Schwarzschild black hole, they take the following asymptotic form in the coordinates t,r,θ,ϕ [Candelas (1980), Candelas *et al.* (1981), Sciama *et al.* (1981)][7]:

$$\langle \phi^2(r) \rangle_B \propto (r - r_g)^{-1}, \tag{10.3.6}$$

$$\langle T^\mu_\nu(r) \rangle_B \propto -\frac{h_s}{2\pi^2(1 - r_g/r)^2} \int_0^\infty \frac{d\omega(\omega^2 + s^2\kappa^2)}{e^{\omega/\theta} - (-1)^{2s}} \,\mathrm{diag}(-1, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}). \tag{10.3.7}$$

In (10.3.7) and hereafter, diag (a, b, c, d) denotes a diagonal matrix with entries equal to a, b, c, d on the diagonal, and h_s is the number of polarizations of the spin-s field.

Quantities characterizing vacuum polarization in this state display such singular behavior because the state itself corresponds, as pointed out above, to a physically nonrealizable situation.

Unfortunately, it was not possible to sum up the series for $\langle \phi^2 \rangle^{\text{ren}}$ and $\langle T_{\mu\nu} \rangle^{\text{ren}}$ obtained after renormalizing the appropriate series of (10.3.3) type for Green's functions and to find explicit final expression for these quantities in the general case. [8] Consequently, either numerical method or the methods of approximate summation of series are used to obtain a result. The work completed up to now has been limited mainly to the case of nonrotating black holes.

Before beginning the representation of known results, we wish to remark that the symmetry of $\langle T^\mu_\nu \rangle$ due to the symmetries of the background Schwarzschild gravitational field and the conservation of $\langle T^\mu_\nu \rangle_{;\mu}$ satisfied by $\langle T^\mu_\nu \rangle$ drastically reduce the number of its independent components. Namely, Christensen and Fulling (1977) demonstrated that any conserved $\langle T^\mu_\nu \rangle$ in the spacetime of a nonrotating black hole admits the following representation:

$$\langle T^\mu_\nu \rangle = \sum_{i=1}^{4} t_i{}^\mu{}_\nu, \tag{10.3.8}$$

where $t_i{}^\mu{}_\nu$ in the coordinates t, r^*, θ, ϕ have the form

$$t_1{}^\mu{}_\nu = \mathrm{diag}\left(-\frac{FH}{r^2} + \frac{1}{2}T, \frac{FH}{r^2}, \frac{1}{4}T, \frac{1}{4}T \right),$$

$$t_2{}^\mu{}_\nu = \mathrm{diag}\left(-\frac{FG}{r^2} - 2\Theta, \frac{FG}{r^2}, \Theta, \Theta \right), \tag{10.3.9}$$

$$t_3{}^\mu{}_\nu = \frac{W}{4\pi r^2} F \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad t_4{}^\mu{}_\nu = \mathrm{diag}\left(\frac{-N}{r^2}F, \frac{N}{r^2}F, 0, 0 \right),$$

and

$$F = (1 - 2M/r)^{-1}, \quad T(r) = \langle T^{\mu}_{\mu}(r)\rangle, \quad \Theta(r) = \langle T^{\theta}_{\theta}(r)\rangle - \frac{1}{4}T(r), \tag{10.3.10}$$

$$H(r) = \frac{1}{2}\int_{2M}^{r} (r' - M)T(r')\,dr', \quad G(r) = 2\int_{2M}^{r} (r' - 3M)\Theta(r')\,dr'.$$

Each of the tensors $t_i{}^{\mu}{}_{\nu}$ satisfies the conservation law $t_i{}^{\mu}{}_{\nu;\mu}=0$. Only $t_1{}^{\mu}{}_{\nu}$ has nonzero trace, only $t_2{}^{\mu}{}_{\nu}$ has trace-free component $t^{\mu}{}_{\nu} - \frac{1}{4}T\delta^{\mu}{}_{\nu}$ whose $\theta\theta$-component is nonzero, only $t_3{}^{\mu}{}_{\nu}$ has nondiagonal components describing fluxes, and only $t_4{}^{\mu}{}_{\nu}$ is not regular on H^+.

In other words, an arbitrary energy-momentum tensor satisfying the conservation law and symmetry conditions imposed by the Schwarzschild metric is characterized unambiguously by fixing two functions $T(r)$ and $\Theta(r)$ (one of them, T, coincides with the trace of the tensor) and two constants W and N: W gives the intensity of radiation of the black hole at infinity ($W = -dM/dt$)) and N vanishes if the energy-momentum tensor is regular on H^+.

The radiation intensity W is not zero only in the Unruh vacuum. The coefficients W for the massless scalar field ($s = 0$), two components neutrino field ($s = 1/2$), electromagnetic field ($s = 1$), and gravitational field ($s = 2$) are [Page (1982), Elster (1983b)]

$$W_0 = 7.4 \times 10^{-5} M^{-2}, \quad W_{1/2} = 8.2 \times 10^{-5} M^{-2},$$

$$W_1 = 3.3 \times 10^{-5} M^{-2}, \quad W_2 = 0.4 \times 10^{-5} M^{-2}. \tag{10.3.11}$$

The coefficients N vanishes for the Unruh and the Hartle–Hawking vacuums.

Numerical calculations published to now cover the case of the scalar massless and electromagnetic fields in the spacetime of the Schwarzschild black hole. The results obtained by Fawcett and Whiting (1982) for $\langle\phi^2\rangle_H$ and $\langle\phi^2\rangle_U$ are plotted in Figure 81. Howard and Candelas (1984) and Howard (1984) calculated the components $\langle T^{\mu}_{\nu}\rangle_H$ for a scalar field and found (and corrected) an error in the earlier

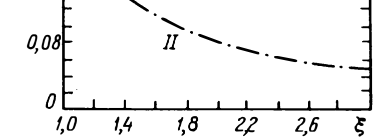


Fig. 81. The values of $(8\pi M)^2\langle\phi^2\rangle$ as functions of $\xi = r/M - 1$: curve I - $\langle\phi^2\rangle_H$, curve II - $\langle\phi^2\rangle_U$.

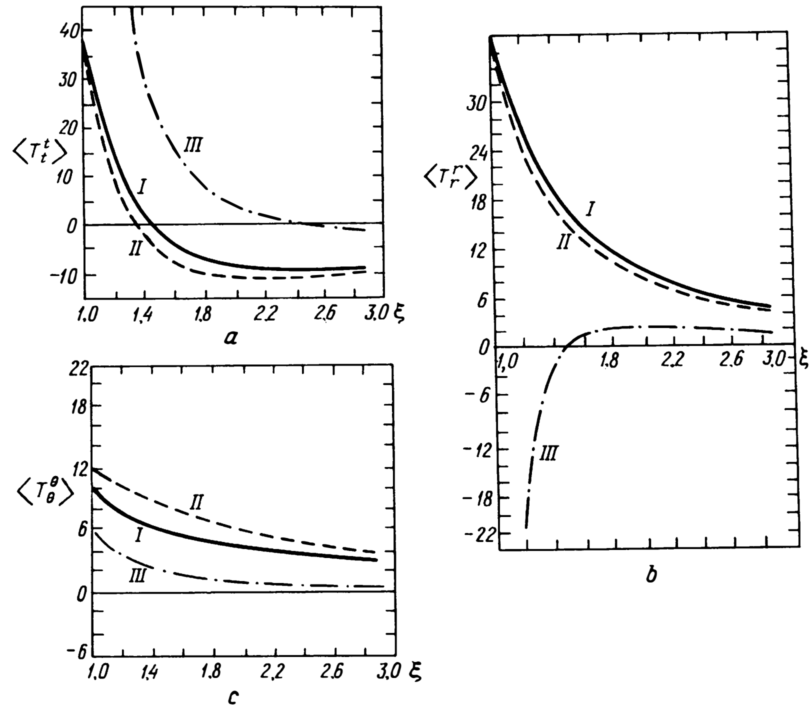


Fig. 82. The values of the components $[90(8\pi M)^4/\pi^2]\langle T^\mu_\nu\rangle$ as functions of ξ. The figures show the behavior of the $\langle T^t_t\rangle$ (a), $\langle T^r_r\rangle$ (b), and $\langle T^\theta_\theta\rangle=\langle T^\phi_\phi\rangle$ (c) components of the energy-momentum tensor. Curves I - components $\langle T^\mu_\nu\rangle_H$, curves II - components $\langle T^\mu_\nu\rangle_H$, curves III - components $\langle T^\mu_\nu\rangle_U$.

calculations of Fawcett (1983). Figure 82 plots nonzero components $\langle T^\mu_\nu\rangle_H$ and the values, calculated by Elster, of $\langle T^\mu_\nu\rangle_U$ for a scalar field.[9] The components of $\langle T^\mu_\nu\rangle$ for the electromagnetic field were calculated by Jensen and Ottewill (1988).

The main feature of the tensor $\langle T^\mu_\nu\rangle_H$ is that its components are finite on the event horizon. Thus, an observer at rest at a point r close to the event horizon, records the energy density $-\langle T^t_t\rangle_H$. This quantity remains finite as $r \to r_g$. On the other hand, the temperature measured by the observer is

$$T_{\text{loc}} = \frac{\kappa}{2\pi}\left(1 - \frac{r_g}{r}\right)^{-1/2}.$$

For example, such measurements of temperature could be realized using a two-level system as a thermometer: Transitions between levels are caused by the absorption and emission of quanta of the field (photons). After a sufficiently long exposure, the probability for a system to occupy the upper level will be less than that for the lower level by a factor $\exp(\Delta E/T_{\text{loc}})$ (ΔE is the energy difference between the levels). Other small-size detectors can be used in a similar manner [Unruh (1976b)]. It is not difficult to show that the temperature in the vicinity of r_g is $T_{\text{loc}} \approx T_a = a/2\pi$, where a is the observer's acceleration; as $r \to r_g, T_{\text{loc}} \to \infty$.[10]

Note, however, that the results plotted in Figure 82 imply that the radiation energy density in the neighborhood of such a point is $\sim\sigma(\kappa/2\pi)^4 \ll \sigma T_a^4$. The law of σT^4 is 'violated' because it fails close to the boundaries where the parameters of the system change greatly over distances on the order of the characteristic wavelength of radiation. Precisely this situation takes place in the case of a black hole close to its surface (at a distance $r - r_g \sim r_g$). The fact that, in contrast to σT_a^4, the quantity $-\langle T_t^t\rangle_H$ is finite at the horizon, can be interpreted as follows: The contribution of vacuum polarization caused by a strong gravitational field close to the horizon compensate exactly for the divergence that would take place for the radiation energy density if the law σT_a^4 were not violated.

Fawcett and Whiting (1982) analyzed the results of computations for scalar massless field ϕ and noticed that the values of $\langle\phi^2\rangle_H$ fit quite well the simple expression

$$\langle\phi^2\rangle_H = \frac{1}{768\pi^2 M^2}\,\frac{1-z^4}{1-z}, \tag{10.3.12}$$

where $z=2M/r$, on the entire interval from r_g to ∞. Using this, Page (1982) suggested a method of approximate computation of $\langle\phi^2\rangle_H$ and $\langle T^\mu_\nu\rangle_H$. The value of $\langle\phi^2\rangle_H$ in Page's approximation, $\langle\phi^2\rangle^P_H$, coincides with (10.3.12). Candelas and Howard (1984), Howard and Candelas (1984), and Howard (1984) have demonstrated that for the scalar massless field the values of $\langle\phi\rangle^P_H$ and $\langle T^\mu_\nu\rangle^P_H$ in Page's approximation fit very well the behavior of $\langle\phi^2\rangle_H$ and $\langle T^\mu_\nu\rangle_H$ (the deviations of $\langle\phi^2\rangle^P_H$ from the true value of $\langle\phi^2\rangle_H$ do not exceed 1% and those of the components $\langle T^\mu_\nu\rangle^P_H$ do not exceed 20%).

Two propositions are basic for constructing Page's approximation:

(1) Let there be two conformal spaces and let the calculation of renormalized expectation values $\langle\phi^2\rangle$ and $\langle T^\mu_\nu\rangle$ be carried out in each of these spaces in the states obtained from each other by the same conformal transformation. Then the following combinations including $\langle\phi^2\rangle$ and $\langle T^\mu_\nu\rangle$ are invariant (i.e., do not depend on which conformal space they were calculated in):

$$Z = g^{1/4}\left[\langle\phi^2\rangle + \frac{1}{288\pi^2}R\right], \tag{10.3.13}$$

$$J^\mu_\nu = g^{1/2}\{\langle T^\mu_\nu\rangle + \alpha_s[(C^{\alpha\mu}{}_{\beta\nu}\ln g)_{;\alpha}{}^{;\beta} + \tfrac{1}{2}R^\beta_\alpha C^{\alpha\mu}{}_{\beta\nu}\ln g] +$$

$$+\beta_s[2H^\mu_\nu - 4R^\beta_\alpha C^{\alpha\mu}{}_{\beta\nu}] + \gamma_s\, I^\mu_\nu\}, \tag{10.3.14}$$

$$H_{\mu\nu} = -R^\alpha_\mu R_{\alpha\nu} + \frac{2}{3}RR_{\mu\nu} + \left(\frac{1}{2}R^\alpha_\beta R^\beta_\alpha - \frac{1}{4}R^2\right)g_{\mu\nu},$$

$$I_{\mu\nu} = 2R_{;\mu\nu} - 2RR_{\mu\nu} + \left(\frac{1}{2}R^2 - 2R_{;\alpha}{}^{;\alpha}\right)g_{\mu\nu}, \tag{10.3.15}$$

$C_{\alpha\beta\gamma\delta}$ is the Weyl tensor (A.4), $R_{\alpha\beta}$ is the Ricci tensor (A.3), R is the scalar curvature, and the coefficients α_s,β_s, and γ_s are related to the coefficients a_s and b_s in the expression for conformal anomalies, (10.1.3), by the formula

$$\alpha_s = a_s, \quad \beta_s = b_s, \quad \gamma_s = \frac{2}{3}a_s. \tag{10.3.16}$$

(2) Let the metric

$$ds^2 = -V^2 dt^2 + h_{ij}\, dx^i\, dx^j \tag{10.3.17}$$

be a static solution of Einstein's vacuum equations ($V^2 = -\xi_{(t)\mu}\xi^{\mu}_{(t)}$, $\xi^{\mu}_{(t)}$ being the Killing vector field). Then there no conformal anomalies in the space with the metric $d\bar{s}^2 = V^{-2}\, ds^2$ [i.e., the expressions in parentheses in the right-hand side of (10.1.3) calculated in this space vanish identically].

Page suggested to calculate $\langle\phi^2\rangle$ and $\langle T^{\mu}_{\nu}\rangle$ first in the space $d\bar{s}^2$, using the solution obtained by WKB approximation for a Green's function in it, and then return to the original physical space taking into account the invariance of the quantities (10.1.13) and (10.3.14). This approach gives for $\langle\phi^2\rangle_H$ of the scalar massless field in the Schwarzschild metric the expression (10.3.12), and for $\langle T^{\mu}_{\nu}\rangle_H$ it gives the following approximate formula

$$\langle T^{\mu}_{\nu}\rangle^{P}_{H} = \frac{\pi^2}{90}\theta^4\left[\frac{1-(4-3z)^2 z^6}{(1-z)^2}(\delta^{\mu}_{\nu} - 4\,\delta^{\mu}_{0}\,\delta^{0}_{\nu}) + 24z^6(3\delta^{\mu}_{0}\,\delta^{0}_{\nu} + \delta^{\mu}_{1}\,\delta^{1}_{\nu})\right], \tag{10.3.18}$$

where $\theta = (8\pi M)^{-1}$ is the black-hole temperature. The behavior of the non-zero components $\langle T^{\mu}_{\nu}\rangle^{P}_{H}$ is shown in Figure 82 (dash-dot line). This approximate expression was used by York (1985) for analyzing the back-reaction of vacuum polarization on the gravitational field of a black hole.

Page's method can also be used for determining the approximate values of $\langle\phi^2\rangle$ and $\langle T^{\mu}_{\nu}\rangle$ in the Boulware vacuum. Calculations give [Frolov and Zel'nikov (1985a)]

$$\langle\phi^2\rangle^{P}_{B} = -\frac{M^2}{48\pi^2 r^4(1-z)}, \tag{10.3.19}$$

$$\langle T^{\mu}_{\nu}\rangle^{P}_{B} = \frac{M^2}{1440\pi^2 r^6}\left[\left(\frac{4-3z}{2(1-z)}\right)^2(-\delta^{\mu}_{\nu} + 4\,\delta^{\mu}_{0}\,\delta^{0}_{\nu}) + 6(3\,\delta^{\mu}_{0}\,\delta^{0}_{\nu} + \delta^{\mu}_{1}\,\delta^{1}_{\nu})\right]. \tag{10.3.20}$$

Expressions (10.3.19)–(10.3.20) agree with asymptotic formulas (10.3.6)–(10.3.7) near the event horizon and manifest correct asymptotic behavior at infinity. It seems

that the accuracy with which (10.3.19) and (10.3.20) reproduce exact values of $\langle\phi^2\rangle_B$ and $\langle T^\mu_\nu\rangle_B$ is of the same order as in the case of the Hartle–Hawking vacuum.

Brown and Ottewill (1985) suggested a slightly different method of determining $\langle\phi^2\rangle$ and $\langle T^\mu_\nu\rangle$ for conformally invariant fields. In the case of a scalar field, the method again yields the expressions, (10.3.18)–(10.3.20), found by Page's approximation. Brown and Ottewill noticed that conformal anomalies vanish not only in the spaces with the metric $d\tilde{s}^2 = V^{-2}\,ds^2$, but also in a wider class of spaces whose metrics are of the form $d\tilde{s}^2=\exp(at)V^{-2}\,ds^2$. If we demand that the choice of state in the space $d\tilde{s}^2$ be such that not only the trace $(\langle \tilde{T}^\mu_\mu \rangle = 0)$ but all other components of $\langle\tilde{T}^\mu_\nu\rangle$ vanish as well, then a quite definite value of $\langle T^\mu_\nu\rangle$ is obtained after we return to the original space. Brown and Ottewill showed that the thus obtained expression for $a = 0$ coincide with $\langle T^\mu_\nu\rangle^P_B$, and that for $a = -2\kappa = -(2M)^{-1}$, it correctly reproduces $\langle T^\mu_\nu\rangle^P_H$. Similar approximate expressions can be obtained in the framework of this approach for the contributions of the neutrino and electromagnetic fields to $\langle T^\mu_\nu\rangle$. A comparison of $\langle T^\mu_\nu\rangle^P_H$ with the results of numerical study for electromagnetic field [Jensen and Ottewill (1988)] shows that the difference becomes large in the vicinity of the horizon. More recently, Frolov and Zel'nikov (1987, 1988) suggested an extended version of the Page approximation which gives a conserved tensor with trace anomaly, and also the correct value of the energy density on the horizon for all massless fields.

The results obtained till now seem to indicate that the main contribution to the vacuum energy-momentum tensor of conformally covariant fields in the black hole fields is made by conformal anomalies. If these anomalies are properly taken into account, the resulting energy-momentum tensor reproduces sufficiently well the exact value of $\langle T^\mu_\nu\rangle$.[11]

We have already mentioned that exact values of $\langle\phi^2\rangle$ and $\langle T^\mu_\nu\rangle$ were successfully calculated in a number of cases where the point of interest lay on the event horizon. This spectacular fact follows from the very special properties of spacetime near the event horizon. We will now discuss these properties in some detail.

The Kerr metric is symmetric under translation in time t and under rotations in ϕ. Let $\xi_{(t)}$ and $\xi_{(\phi)}$ be the corresponding Killing vector fields, and $\eta = \xi_{(t)} + \Omega^H\xi_{(\phi)}$ be their linear combination which is tangent to the generators of the horizon. It is readily shown that $\xi_{(\phi)}$ vanishes on the symmetry axis ($\theta = 0$, $\theta = \pi$), while η vanishes on the two-dimensional surface S of intersection of the horizon H^- and H^+ (the surface of horizon bifurcation). The antisymmetric tensors $\xi_{(\phi)\mu;\nu}|_{\xi(\phi)=0}$ and $\eta_{\mu;\nu}|_{\eta=0}$ are nondegenerate.[12] Obviously, the polar ($\theta = 0$ and $\theta = \pi$) points x_0 of the surface S remain fixed under shifts both in t and ϕ.

If we stipulate that $\langle T^\mu_\nu\rangle$ in the chosen state has the same symmetric properties as the background physical spacetime, this quantity must satisfy the equation

$$\mathcal{L}_\xi\langle T_{\mu\nu}\rangle \equiv \xi^\alpha\langle T_{\mu\nu}\rangle_{;\alpha} + \xi^\alpha{}_{;\mu}\langle T_{\alpha\nu}\rangle + \xi^\alpha{}_{;\nu}\langle T_{\mu\alpha}\rangle = 0, \qquad (10.3.21)$$

where ξ is the Killing vector field and $\mathcal{L}_\xi$ is the Lie derivative along it. At points where $\xi^\alpha = 0$, these equations turn into constraints on the algebraic structure of

$\langle T_{\mu\nu}\rangle$. It can be shown, when solving these equation, that such a regular vacuum energy-momentum tensor has the following form at the pole x_0 of the surface S:

$$\langle T_{\mu\nu}\rangle = A(k_\mu l_\nu + l_\mu k_\nu) + B(m_\mu \overline{m}_\nu + \overline{m}_\mu m_\nu). \tag{10.3.22}$$

[Frolov and Zel'nikov (1985b)]. Here $z_a = (k, l, m, \overline{m})$ are the vectors of the complex null tetrad:

$$k_\alpha \, dx^\alpha = -\, dt + \frac{\rho^2}{\Delta} dr + a \sin^2 \theta \, d\phi,$$

$$l_\alpha dx^\alpha = \frac{\Delta}{2\rho^2}\left(-dt - \frac{\rho^2}{\Delta} r + a \sin^2 \theta \, d\phi \right), \tag{10.3.23}$$

$$m_\alpha \, dx^\alpha \equiv \overline{(\overline{m}_\alpha \, dx^\alpha)}$$

$$= \frac{1}{\sqrt{2(r + ia \cos \theta)}} [-ia \sin \theta \, dt + \rho^2 \, d\theta + i(r^2 + a^2) \sin \theta \, d\phi].$$

In other words, this tensor is determined by two constants A and B, the difference between them being fixed by the value of the conformal anomalies

$$2(B - A) = \langle T^\mu_\mu\rangle = (a_s + b_s) C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta}. \tag{10.3.24}$$

In the case of a spherically symmetric black hole, any point of the surface S can be chosen for x_0. The nonzero components of the energy-momentum tensor then take the form

$$T^t_t = T^r_r = -A, \qquad T^\theta_\theta = T^\phi_\phi = B. \tag{10.3.25}$$

The continuity of $\langle T_{\mu\nu}\rangle$ and invariance (10.3.21) imply that $\langle T_{\mu\nu}\rangle$ is of a form similar to (10.3.22) at the pole of the event horizon and outside the surface S.

Another factor also leads to substantial simplification of the problem of calculating $\langle \phi^2\rangle$ and $\langle T^\mu_\nu\rangle$ at the event horizon. Let $\Pi(x,x')$ be an arbitrary biscalar [e.g., Green's function $G_H(x,x')$ of a scalar field] with the symmetry properties of the background spacetime. Then

$$\mathfrak{L}_\xi \Pi(x, x') + \mathfrak{L}_{\xi'} \Pi(x, x')$$
$$= \left(\xi^\alpha \frac{\partial}{\partial x^\alpha} + \xi^{\alpha'} \frac{\partial}{\partial x'^{\alpha'}} \right) \Pi(x, x') = 0. \tag{10.3.26}$$

Here $\mathfrak{L}_\xi (\mathfrak{L}_{\xi'})$ is the Lie derivative along the Killing vector field ξ with respect to the first (second) argument. If the points x and x' do not coincide with the rotation axis and do not lie on the surface S, Equation (10.3.26) shows that the biscalar $\Pi(x, x')$ is a function of the differences $t - t'$ and $\phi - \phi'$. If the point x' lies on the

pole of the surface S, the function $\Pi(x, x_0)$ is completely independent of either t or ϕ.

This results, among other things, in a substantial simplification of the wave equation that governs the function $-iG_H(x,x_0)$, and an exact solution can be obtained. In the coordinates $R = \Delta^{1/2} \sin\theta$, $z = (r - M)\cos\theta$ this equation in the Kerr metric becomes

$$\left[\frac{1}{R}\partial_R(R\,\partial_R) + \partial_z^2\right](-iG_H(R, z; x_0)) = -\frac{\kappa}{4\pi^2 R}\delta(R)\,\delta(z - z_0), \tag{10.3.27}$$

where $z_0 = \sqrt{M^2 - a^2}$. Note that the solution of this equation coincides with the potential of the field of a point-like charge $e = \kappa/8\pi^2$ placed at a point $z = z_0$ of the axis $x = y = 0$ ($R^2 = x^2 + y^2$) in a flat space. Using this solution, we obtain

$$G_H(x, x_0) = \frac{i\kappa}{8\pi^2(r - M - \sqrt{M^2 - a^2}\cos\theta)}. \tag{10.3.28}$$

Subtracting the divergent part from this expression [see (10.1.10)] and tending x to x_0, we obtain for $\langle\phi^2(x_0)\rangle_H$ at the pole of the horizon the following value [Frolov (1982, 1983c)]:[13]

$$\langle\phi^2(x_0)\rangle_H = \frac{1}{48\pi^2}\frac{r_+^2 - 3a^2}{(r_+^2 + a^2)^2}. \tag{10.3.29}$$

[Candelas (1980) derived this expression for $a = 0$ by summing up the series representation of Green's function.]

Frolov and Zel'nikov (1985b) used a similar method for calculating $\langle T^\mu_\nu(x_0)\rangle_H$ for the electromagnetic field. After correction of the technical mistake found by Jensen, McLaughlin, and Ottewill (1988), this method gives for the coefficients A and B in (10.3.22) the expressions

$$A = \frac{1}{1920\pi^2 M^4 r_+^3}[-15M^3 - 75M^2(M^2 - a^2)^{1/2} + 52(M^2 - a^2)^{3/2}],$$

$$B = \frac{1}{1920\pi^2 M^4 r_+^3}[-15M^3 + 3M^2(M^2 - a^2)^{1/2} - 52(M^2 - a^2)^{3/2}]. \tag{10.3.30}$$

Let us look at one interesting property of expression (10.3.29). It is easily verified that

$$\langle\phi^2(x_0)\rangle_H = \frac{1}{48\pi^2}K, \tag{10.3.31}$$

where K is the Gaussian curvature of the two-dimensional surface of the black hole at its pole x_0. [For the calculation of K on the surface of a Kerr black hole, see

Smarr (1973b).] Expression (10.3.31) is also valid in the case of a black hole placed in an external static axially symmetric gravitational field.

It is easily proved that in Page's approximation, the quantity $\langle\phi^2\rangle_H^P$ at the event horizon of any static black hole (including black holes distorted by an external field) can be written in the form (10.3.31). The quantity $\varepsilon = -\langle T^t_t\rangle_H^P$ characterizing the energy density on the black-hole surface, is given in this approximation by the expression [Frolov and Sanchez (1985)]

$$\varepsilon = -(7a_s + 12b_s)K^2 + a_s\, {}^{(2)}\Delta K, \tag{10.3.32}$$

where ${}^{(2)}\Delta$ is the two-dimensional Laplacian on the black-hole surface, and a_s and b_s are the coefficients in the expression for conformal anomalies, (10.1.3).

The contribution of massive fields to vacuum polarization near black holes of a mass greater than the Planck mass is much smaller (by a factor $\varepsilon = (\lambda/L)^2$, $\lambda = \hbar/mc$, L denoting the characteristic curvature radius) than that of massless fields. Nevertheless, an analysis of the contribution of massive fields is instructive for a number of reasons. Note, first of all, that if $\varepsilon \ll 1$, it is possible to separate the contributions of real and virtual particles to $\langle T^\mu_\nu\rangle$. Thus, the contribution of real particles of the heat bath to $\langle T^\mu_\nu\rangle_H$ in the Hartle–Hawking vacuum state contains a small factor $\exp(-\beta GMm/\hbar c)$ while the contribution of virtual particles is a power function of M.[14] A much more detailed study of the effect of vacuum polarization of massive fields is possible because one can make use of the expansion in a small dimensional parameter ε. If the background gravitational field satisfies Einstein's vacuum equations, the quantity $\langle T^\mu_\nu\rangle_H$ for the massive scalar, spinor, and vector fields can be obtained in the first nonvanishing order in ε by varying the following effective action [Frolov and Zel'nikov (1984)]:

$$\langle T_{\mu\nu}(x)\rangle^{(s)}_H = -\frac{2}{\sqrt{-g}}\frac{\delta W^{(s)}}{\delta g^{\mu\nu}(x)}, \tag{10.3.33}$$

where

$$W^{(s)} = (96\times 7!\,\pi^2 m^2)^{-1}\int d^4x\sqrt{-g}\,(A_s R_{\alpha\beta}{}^{\gamma\delta}R_{\gamma\delta}{}^{\varepsilon\zeta}R_{\varepsilon\zeta}{}^{\alpha\beta} +$$

$$+ B_s R R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}). \tag{10.3.34}$$

Here m is the mass of the field, and the coefficients A_s and B_s for the spin-s field are

$$A_0 = 1,\quad A_{1/2} = -4,\quad A_1 = 3,$$

$$B_0 = 18 - 84\xi,\quad B_{1/2} = 12,\quad B_1 = -30, \tag{10.3.35}$$

respectively. (ξ is the coefficient with the term $R\phi^2$ in the action for the scalar field; the field is conformally invariant if $\xi = \frac{1}{6}$ and $m = 0$). In this approximation, $\langle T^\mu_\nu\rangle^{(s)}_H$ for a Schwarzschild black hole is of the form [Frolov and Zel'nikov (1982)]

$$\langle T^{\mu}_{\nu}\rangle^{(s)}_{H} = \frac{M^2}{10080\pi^2 m^2 r^8}\,[A_s p^{\mu}_{\nu} + 3(2B_s + A_s) q^{\mu}_{\nu}], \tag{10.3.36}$$

where nonzero components p^{μ}_{ν} and q^{μ}_{ν} (for $z = 2M/r$) are

$$p^t_t = -15 + 16z,\ p^r_r = -3 + 4z,\ p^{\theta}_{\theta} = p^{\phi}_{\phi} = 9 - 11z,$$

$$q^t_t = -10 + 11z,\ q^r_r = 4 - 3z,\ q^{\theta}_{\theta} = q^{\phi}_{\phi} = -12 + 14z. \tag{10.3.37}$$

Note the qualitative similarity in the behavior of $\langle T^{\mu}_{\nu}\rangle_H$ of massless and massive fields. The energy density $\varepsilon = -\langle T^t_t\rangle_H$ of a scalar massive ($\xi = \frac{1}{6}$) field is positive far from the black hole but reverses the sign and becomes negative at the horizon. A similar effect occurs for a conformal scalar massless field. In the case of a massive vector field, the energy density ε is positive at the event horizon.

The rotation of the black hole produces a circular energy density flux in the surrounding space, due to vacuum polarization. This flux is described by the component $\langle T^t_{\phi}\rangle \neq 0$. The expression for $\langle T^{\mu}_{\nu}\rangle^{(s)}$ of massive fields in the Kerr metric was derived and analyzed by Frolov and Zel'nikov (1983, 1984).

To complete the exposition of the main results concerning $\langle\phi^2\rangle$ and $\langle T^{\mu}_{\nu}\rangle$, we give Table IV containing a list of the main papers treating vacuum polarization in black holes with a brief characterization of the obtained results. The columns of the table show the mass m and spin s of the field for which calculations were carried out, indicate whether the black hole was assumed to be rotating ($J \neq 0$) or charged ($Q \neq 0$), show what quantities ($\langle\phi^2\rangle$ or $\langle T^{\mu}_{\nu}\rangle$) for which state ($|H\rangle$, $|H, r_0\rangle$, $|U\rangle$ or $|B\rangle$) and where (at the horizon or beyond it) where calculated, and give brief information on the method by which the calculations were conducted.

It conclusion, it should be noted that not only has a qualitative understanding of the specifics of vacuum polarization effects in black holes been achieved by now, but so has a good quantitative description as well.

Table IV

References	Field Parameters: Mass m	Field Parameters: Spin s	Black hole parameters: Angular momentum J	Black hole parameters: Charge Q	Calculation conducted: on event horizon	Calculation conducted: outside horizon	$\langle\phi^2\rangle$	$\langle T^{\mu}_{\nu}\rangle$	Vacuum states	Remarks
Christensen & Fulling (1977)			0	0	+	+		+	H, B, U	General analysis
Candelas (1980)	0	0	0	0	+	+	+	+	H, B, U	Series representation for G, $\langle\phi^2\rangle$ and $\langle T^{\mu}_{\nu}\rangle$.
Candelas *et al.* (1981)	0	1; 2	$\neq 0$	0	+			+	B	Summation of series
Fawcett & Whiting (1982)	0	0	0	0	+	+	+		H, B, U	Numerical analysis
Page (1982)	0	0	0	0	+	+	+	+	H	Approximate method
Nugaev (1982)	0	0; 1	0	0	+			+	B	Approximate method
Frolov (1982)	0	0	$\neq 0$	$\neq 0$	+		+		H	Explicit analytic expression
Elster (1982a)	0	0	0	0	+	+		+	$(H; r_0)$	Analysis of boundary effects
Elster (1982b)	0	0	0	0	+		+		$(H; r_0)$	Explicit expression
Frolov & Zel'nikov (1982)	$\neq 0$	0	0	0	+	+		+	H	Expansion in the parameter $\varepsilon=(m_{\text{Pl}}^2/Mm)^2$
Fawcett (1983)	0	0	0	0	+	+		+	H	Numerical analysis
Elster (1983a)	0	0	0	0	+			+	$(H; r_0)$	Numerical analysis

Frolov (1985)	0	0	0	0	+	+	+	+	H	Approximate method
Frolov & Zel'nikov (1983)	$\neq 0$	$0; \frac{1}{2}, 1$	$\neq 0$	0	+	+		+	H	Expansion in the parameter ε
Frolov & Garcia (1983)	0	0	0	0	+		+		H	Perturbed black hole. Explicit expression
Frolov & Zel'nikov (1984)	$\neq 0$	$0; \frac{1}{2}, 1$	$\neq 0$	0	+	+		+	H	Expansion in the parameter ε
Candelas & Howard (1984)	0	0	0	0	+	+	+		H	Numerical analysis
Howard & Candelas (1984)	0	0	0	0	+	+		+	H	Justification and evaluation of accuracy of Page's approximation
Howard (1984)	0	0	0	0	+	+		+	H	
Zannias (1984)	0	0	0	$\neq 0$	+	+		+	H	Approximate method
Elster (1984)	0	1	0	0	+			+	H	Explicit expression
Frolov & Sanchez (1985)	0	$0; \frac{1}{2}, 1$	0	0	+		+	+	H, B	Perturbed black hole Approximate method
Brown & Ottewill (1985)	0	$0; \frac{1}{2}, 1$	0	0	+	+		+	H, B	Approximate method
Frolov & Zel'nikov (1985b)	0	1	$\neq 0$	0	+			+	H	Explicit expression
Brown *et al.* (1986)	0	$0; \frac{1}{2}, 1$	0	0	+	+		+	H	Approximate method
Frolov & Zel'nikov (1987, 1988)	0	$0; \frac{1}{2}, 1$	0	0	+	+		+	H	Approximate method
Jensen *et al.* (1988)	0	1	0	0	+			+	H	Explicit expression
Jensen and Ottewill (1988)	0	1	0	0	+	+		+	H	Numerical analysis

Chapter 11

Thermodynamics of Black Holes

11.1 Black Holes and Thermodynamics

The discovery of the thermal radiation of black holes by Hawking was a complete surprise to most specialists, even though quite a few indications of a close relationship between black-hole physics and thermodynamics had crystallized before this discovery.

Wheeler seems to have been the first to notice that the existence of a black hole in the classical theory of gravitation contradicts the law of nondecreasing entropy. Indeed, imagine that a black hole absorbs a hot body possessing a certain amount of entropy. Then the observer outside it finds that the total entropy is the part of the world accessible to his observation has decreased. This disappearance of entropy could be avoided in a purely formal way if we simply attributed the entropy of the absorbed body to the inner region of the black hole. In fact, this 'solution' is patently unsatisfactory because any attempt of an 'outside' observer to determine the amount of entropy 'absorbed' by the black hole, is doomed to failure. Quite soon after the event, the black hole becomes stationary and completely 'forgets', as a result of 'balding' such 'details' as the structure of the absorbed body and its entropy.

If we are not inclined to forego the law of nondecreasing entropy because a black hole has formed somewhere in the Universe, we have to conclude that any black hole *as such* possesses a certain amount of entropy and that a hot body falling into it not only transfers to it the mass, angular momentum, and electric charge, but also the entropy S as well. As a result, the entropy of the black hole increases by at least S. Bekenstein (1973a) noticed that the properties of one of the black-hole characteristics – its area A – resemble those of entropy. Indeed, the Hawking theorem implies that the area A does not decrease in any classical processes, that is, it behaves as entropy does. It was found, in fact, that the analogy of black-hole physics to thermodynamics reaches quite far. It covers both specific thermodynamic devices (like the heat engine) and the general laws of thermodynamics, each of which has an analogue in black-hole physics.

An arbitrary black hole, like a thermodynamic system, reaches an equilibrium (stationary) state after the relaxation processes are completed. In this state, it is

completely described by fixing a small number of parameters: M, J, Q. The area A of a stationary black hole is a function of these parameters.

$$A = 4\pi(2M^2 - Q^2 + 2M\sqrt{M^2 - Q^2 - J^2/M^2}). \tag{11.1.1}$$

This relation can be reversed, yielding the formula for the internal energy of a black hole

$$M \equiv M(A, J, Q) = \left\{\frac{\pi[(Q^2 + A/4\pi)^2 + 4J^2]}{A}\right\}^{1/2}. \tag{11.1.2}$$

The internal energies of two stationary black holes with slightly different values of area (by δA), angular momentum (by δA), and electric charge (by δQ), differ by an amount

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega^H \delta J + \Phi^H \delta Q, \tag{11.1.3}$$

where $\kappa = 4\pi\sqrt{M^2 - Q^2 - J^2/M^2}/A$ is the surface gravity, $\Omega^H = 4\pi J/MA$ is the angular velocity, and $\phi^H = 4\pi Q r_+/A$ is the electric potential of the black hole. The second and third terms of this formula describe the changes in rotational energy and electric energy, respectively.

This relation is similar to the first law of thermodynamics. An analogue of temperature (the variable conjugate to entropy) is a quantity proportional to surface gravity κ. Hawking's result on the thermal nature of the radiation of stationary black holes not only supports this analogy, but also supplies the coefficient relating temperature θ to surface gravity κ

$$\theta = \hbar\kappa/2\pi ck. \tag{11.1.4}$$

Note that relation (11.1.3) is identical to the first law of thermodynamics,

$$\delta E = \theta\,\delta S^H + \Omega^H \delta J + \Phi^H \delta Q, \tag{11.1.5}$$

if one assumes the following expression for the entropy of the black hole

$$S^H = A/4l_{\rm Pl}^2, \quad l_{\rm Pl}^2 = \hbar G/c^3. \tag{11.1.6}$$

These arguments give sufficient ground for treating the black-hole physics-thermodynamics analogy quite seriously. In Section 11.3 we will discuss the main laws of black-hole physics that play a role similar to the laws of thermodynamics, but first we analyze the general properties of surface gravity κ and derive the so-called mass formulas that generalize relations (11.1.2) and (11.1.3) to the case of arbitrary stationary black holes surrounded by a stationary distribution of matter and fields.

11.2. Surface Gravity. Mass Formula

According to uniqueness theorems (Sections 6.3 and 6.4), a lone stationary black hole is generally a Kerr–Newman one. The angular velocity Ω^H, surface gravity κ, and electric potential ϕ^H of such black holes are constant on the event horizon. This property is still preserved if the black hole is surrounded by matter, provided the spacetime geometry remains stationary and axially symmetric or static. Let us go into the details of this property because it is essential for elaborating the thermodynamics analogy in black-hole physics.

Let $\xi^\mu_{(t)}\,\partial_\mu = \partial_t$ and $\xi^\mu_{(\phi)}\partial_\mu = \partial_\phi$ be the Killing vector fields corresponding to time translation and to rotation. Define a bivector $\rho_{\mu\nu}$:

$$\rho_{\mu\nu} = 2\xi_{(t)[\mu}\xi_{(\phi)\nu]}. \tag{11.2.1}$$

Carter (1973a) demonstrated that if the circularity condition is met (see Section 6.4), the event horizon of an arbitrary stationary axially symmetric black hole coincides with the set of points at which the bivector $\rho_{\mu\nu}$ becomes light-like ($\rho_{\mu\nu}\rho^{\mu\nu} = 0$). The vectors l^μ tangent to the event horizon coincide in direction with the null vector that lies in the two-dimensional null area element defined by the vectors $\xi_{(t)}$ and $\xi_{(\phi)}$. Choosing the normalization of l^μ in an appropriate manner, we find

$$l^\mu = \xi^\mu_{(t)} + \Omega^H\xi^\mu_{(\phi)}. \tag{11.2.2}$$

As follows from spacetime symmetry, the angular velocity Ω^H of the black hole cannot be a function of either time t or angle ϕ:

$$\xi^\mu_{(t)}\Omega^H_{,\mu} = \xi^\mu_{(\phi)}\Omega^H_{,\mu} = 0. \tag{11.2.3}$$

It is not trivial that Ω^H is also independent of the 'latitude' of a point on the black-hole surface, that is, it is constant everywhere on this surface. To prove this proposition, we introduce the notation

$$X = \xi_{(\phi)}\cdot\xi_{(\phi)},\quad W = \xi_{(t)}\cdot\xi_{(\phi)},\quad V = -\xi_{(t)}\cdot\xi_{(t)}. \tag{11.2.4}$$

Since $\xi_{(\phi)}$ (as well as $\xi_{(t)}$) lies in the plane tangent to the event horizon, $\xi_{(\phi)}l = 0$ and we find from (11.2.2) that

$$X\Omega^H = -W. \tag{11.2.5}$$

If this relation is differentiated with respect to x^α and then the commutativity $[\xi_{(t)},\xi_{(\phi)}] = 0$ is used, it is not difficult to obtain the following equality:

$$X^2\Omega^H_{,\alpha} = 2(\xi^\beta_{(\phi)}W - \xi^\beta_{(t)}X)\xi_{(\phi)\beta;\alpha}. \tag{11.2.6}$$

Now we multiply both sides of this equality by $\rho_{\gamma\delta}$ and antisymmetrize it with respect to the indices α,γ and δ. If we also take into account the relation

$$\xi_{(\phi)\alpha;[\beta}\rho_{\gamma\delta]} = \xi_{(\phi)\alpha}\xi_{(\phi)[\beta;\gamma}\xi_{(t)\delta]} - \xi_{(t)\alpha}\xi_{(\phi)[\beta;\gamma}\xi_{(\phi)\delta]}, \tag{11.2.7}$$

implied by circularity condition (6.4.4), and recall that the invariant $W^2 + XY = -2\rho_{\alpha\beta}\rho^{\alpha\beta}$ vanishes on the horizon, it can be shown that the right-hand side of the obtained expression equals zero, so that

$$X^2 \Omega^H_{[,\alpha}\rho_{\gamma\delta]} = 0. \tag{11.2.8}$$

Taken off the symmetry axis ($X \neq 0$), this condition implies that $\Omega^H_{,\alpha}$ lies in the two-dimensional plane in which lie the vectors $\xi_{(t)}$ and $\xi_{(\phi)}$ and relation (11.2.3) shows that $\Omega^H_{,\alpha} = 0$. The constancy of Ω^H on the event horizon is thereby proved. (At the poles, Ω^H is found from the continuity.)

A somewhat different formulation can be given to the constancy of the angular velocity Ω^H of the black hole. We introduce a Killing vector field η

$$\eta^\mu = \xi^\mu_{(t)} + \Omega^H \xi^{\ \mu}_{(\phi)}. \tag{11.2.9}$$

This vector field is tangent at the event horizon to the generators of the horizon, and $\eta \cdot \eta = 0$.[1] In other words, the constancy of Ω^H implies that the event horizon coincides with the *Killing horizon.* This horizon is defined as the null surface whose null tangent vectors coincide (after an appropriate normalization) with the vectors of some fixed Killing field.

Let us prove now the constancy of the surface gravity κ. We dealt with this quantity on a number of occasions, for example, when discussing various properties of the Kerr black hole. Now we will give a more general definition of this quantity, valid for an arbitrary stationary (not necessarily lone) black hole.

Since the quantity $\eta \cdot \eta$ (equal to zero) is constant on the surface of the event horizon, the vector $(\eta . \eta)_{;\nu}$ is normal to this surface. This being a null surface, we have

$$(\eta \cdot \eta)^{;\nu} = 2\kappa\eta^\nu, \tag{11.2.10}$$

where κ is the invariant function on the black-hole surface, called the surface gravity. Since $\xi_{(t)}$ and $\xi_{(\phi)}$ are tangent to the surface and $\mathcal{L}_{\xi_{(t)}}\eta = \mathcal{L}_{\xi_{(\phi)}}\eta = 0$, we can apply the operators $\mathcal{L}_{\xi_{(t)}}$ and $\mathcal{L}_{\xi_{(\phi)}}$ to both sides of (11.2.10) and verify that $\xi^\mu_{(t)}\kappa_{,\mu} = \xi^\mu_{(\phi)}\kappa_{,\mu} = 0$. This property is a simple reflection of the symmetry inherent in spacetime. The independence of κ of the 'latitude' of a point on the surface of the black hole is a considerably less trivial conclusion.

Following Bardeen *et al.* (1973), it is convenient to use the tetrad formalism for proving the constancy of κ on the entire event horizon. Correspondingly, we supplement $l^\mu = \eta^\mu|_{H+}$ to the complex null tetrad by choosing complex null vectors m^μ and $\overline{m}^\mu$ tangent to the horizon surface and normalized by the relation $m^\mu \overline{m}^\mu = 1$, and a real null vector n^μ orthogonal to m^μ and $\overline{m}^\mu$ and normalized by the condition $l^\mu n_\mu = -1$. Using the complex null tetrad introduced in this way, we can write κ in the following form:

$$\kappa = -l_{\nu;\mu} n^\nu l^\mu. \tag{11.2.11}$$

Indeed, if we use $\eta_{\mu;\nu} = -\eta_{\nu;\mu}$, rewrite (11.2.10) in the form

$$\eta^{\mu}\eta^{\nu}_{;\mu} = \kappa\eta^{\nu}, \tag{11.2.12}$$

and recall that $\eta^{\mu} = l^{\mu}$ on H, then (11.2.11) is obtained by multiplying (11.2.12) by n^{ν}.

Using (11.2.11), we obtain

$$m^{\alpha}\kappa_{;\alpha} = -l_{\nu;\mu\alpha}n^{\nu}l^{\mu}m^{\alpha} - l_{\nu;\mu}n^{\nu}{}_{;\alpha}l^{\mu}m^{\alpha} - l_{\nu;\mu}n^{\nu}l^{\mu}{}_{;\alpha}m^{\alpha}. \tag{11.2.13}$$

The first of the terms in the right-hand side depends only on the value of l^{μ} on H; hence, we can make use of the fact that l^{μ} on this surface coincides with the Killing vector field η^{μ}. Thus, we can use relation (A.15) for the Killing vector field $\eta_{\nu;\mu\alpha} = R_{\beta\alpha\mu\nu}\eta^{\beta}$ to recast the first term in the right-hand side of (11.2.13) to the form $-R_{\alpha\beta\gamma\delta}l^{\alpha}m^{\beta}l^{\gamma}n^{\delta}$. Using (11.2.12) and the relation $l_{\nu}n^{\nu}{}_{;\alpha} = -n^{\nu}l_{\nu;\alpha}$, we rewrite the second term in the right-hand side of (11.2.13) in the form $\kappa l_{\nu;\alpha}n^{\nu}m^{\alpha}$. Let us now show that this term and the last one in the right-hand side of (11.2.13) cancel out. First we remark that the conditions of normalization of the null tetrad yield the relation

$$g^{\mu\beta} = -n^{\beta}l^{\mu} - l^{\beta}n^{\mu} + m^{\beta}\bar{m}^{\mu} + \bar{m}^{\beta}m^{\mu}. \tag{11.2.14}$$

Making use of this relation and the conditions of no shear or expansion of the event horizon surface,

$$\rho = -l_{\alpha;\beta}m^{\alpha}\bar{m}^{\beta} = 0, \quad \sigma = -l_{\alpha;\beta}m^{\alpha}m^{\beta} = 0, \tag{11.2.15}$$

we rewrite the last term in the right-hand side of (11.2.13) in the form

$$\begin{aligned} -l_{\nu;\mu}n^{\nu}l^{\mu}{}_{;\alpha}m^{\alpha} &= -l_{\nu;\mu}n^{\nu}g^{\mu\beta}l_{\beta;\alpha}m^{\alpha} \\ &= l_{\nu;\mu}n^{\nu}l^{\mu}n^{\beta}l_{\beta;\alpha}m^{\alpha} = -\kappa l_{\nu;\alpha}n^{\nu}m^{\alpha}. \end{aligned} \tag{11.2.16}$$

This expression differs from the second term only in sign and the two cancel out. As a result, we have

$$m^{\alpha}\kappa_{;\alpha} = -R_{\alpha\beta\gamma\delta}l^{\alpha}m^{\beta}l^{\gamma}n^{\delta}. \tag{11.2.17}$$

Note that on the horizon surface

$$l_{\alpha;\beta}m^{\alpha}{}_{;\gamma}\bar{m}^{\beta}m^{\gamma} = l_{\alpha;\beta}m^{\alpha}\bar{m}^{\beta}{}_{;\gamma}m^{\gamma} = 0. \tag{11.2.18}$$

This can be confirmed using (11.2.14), (11.2.15), and the normalization conditions for the tetrad vectors. Therefore, in view of (A.15) and (11.2.15), we find

$$0 = -\rho_{;\gamma}m^{\gamma} = (l_{\alpha;\beta}m^{\alpha}\bar{m}^{\beta})_{;\gamma}m^{\gamma} = R_{\varepsilon\alpha\beta\gamma}l^{\varepsilon}m^{\alpha}\bar{m}^{\beta}m^{\gamma}$$

$$= -R_{\alpha\beta} l^\alpha m^\beta + R_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma n^\delta. \tag{11.2.19}$$

This relation and Einstein's equation $R_{\alpha\beta} l^\alpha m^\beta = 8\pi T_{\alpha\beta} l^\alpha m^\beta$ allow the rewriting of (11.2.17) in the form

$$m^\alpha \kappa_{;\alpha} = -8\pi T_{\alpha\beta} l^\alpha m^\beta. \tag{11.2.20}$$

To complete the proof of the constancy of κ, assume that the energy-momentum tensor $T_{\alpha\beta}$ satisfies the energy dominance condition (see Appendix), that is, $T_{\alpha\beta} l^\alpha$ is a nonspace-like vector for the light-like vector l^α. If this condition is satisfied, the vector $T_{\alpha\beta} l^\alpha$ has to be light-like on the event horizon [the case of a time-like vector is excluded because at the horizon $T_{\alpha\beta} l^\alpha l^\beta = \Phi = 0$; see (6.2.2)]. Hence, $T_{\alpha\beta} l^\alpha m^\beta = 0$ and relation (11.2.20) proves that κ is constant at the horizon.

The integral curves $x^\mu = x^\mu(v)$ of the vector field η^μ ($dx^\mu/dv = \eta^\mu$) at the event horizon coincide with its generators; hence, they are geodesics. This can also be shown directly, by using relation (11.2.12). As the right-hand side of this equation does not vanish, the Killing parameter v does not coincide with the affine parameter λ along the null geodesic described by this equation. The relation between λ and v is of the form $\lambda = ae^{\kappa v} + b$, where a and b are arbitrary constants corresponding to the arbitrariness in the choice of the affine parameter λ.

Let us dwell on the physical interpretation of κ. Consider a stationary observer moving near a black hole, such that the world line of this observer coincides with the integral curve of the Killing vector field η^μ. The velocity four-vector of this observer is

$$u^\mu = \frac{\eta^\mu}{|\eta_\alpha \eta^\alpha|^{1/2}}.$$

If η^μ is described by (11.2.9), this observer, located near the event horizon, rotates at an angular velocity equal to that of the black hole. Obviously, the motion of this observer is nongeodesic, its four-acceleration a^μ [cf. (A.16)] being equal to

$$a^\mu = -\frac{\eta^\mu_{;\alpha} \eta^\alpha}{\eta_\beta \eta^\beta}. \tag{11.2.21}$$

We denote $a = |a_\mu a^\mu|^{1/2}$ and $\eta = |\eta_\beta \eta^\beta|^{1/2}$. Comparing now (11.2.12) and (11.2.21), we obtain

$$\kappa = \lim(\eta a), \tag{11.2.22}$$

where lim indicates taking the limit in which the point where the expression ηa is calculated tends to the event horizon.

For a nonrotating black hole, η is nothing other than the factor of red shift ($\eta = \sqrt{-g_{tt}}$). Imagine that a body is at rest close to the event horizon and suspended on a weightless rigid string. If the mass of the body is m, it will be at rest if the string applies to it a force f^μ such that $f = |f_\mu f^\mu|^{1/2} = ma$. It can be shown that it is sufficient to apply the force $f_0 = m\eta a$ to the other (distant) end of the string (see

Section 2.2). The quantity ηa can be interpreted as the acceleration of a body at rest in the vicinity of the black hole, as measured in the reference frame of a distant observer. In other words, the surface gravity κ characterizes the limiting 'strength' of the gravitational field on the black hole surface, evaluated by a distant observer. If the black hole is rotating, κ has a similar meaning, the only difference being that the body rotates at the angular velocity of the black hole.

In describing the physical effects in the field of a charged black hole, expressions include, in addition to κ, another invariant quantity ϕ^H, that is, the electric field potential on the black hole surface. In Chapter 7 we have mentioned that this quantity is constant on the surface of Kerr–Newman black holes. Let us show that this result is general and valid for any static or axisymmetric stationary (not necessarily lone) black hole.

Let ξ^μ be a Killing field and $F_{\mu\nu}$ be the electromagnetic field tensor satisfying Maxwell's equations

$$F^{\mu\nu}{}_{;\nu} = 4\pi j^\mu,$$

$$F_{[\mu\nu,\alpha]} = 0, \tag{11.2.23}$$

and obeying the symmetry condition

$$\mathcal{L}_\xi F_{\mu\nu} = 0. \tag{11.2.24}$$

Then it is readily shown that the vector $E_\mu = -F_{\mu\nu}\xi^\nu$ satisfies the condition

$$E_{[\mu;\nu]} = 0 \tag{11.2.25}$$

and, therefore, is the gradient of a function Φ

$$E_\mu = \Phi_{,\mu}. \tag{11.2.26}$$

Let us show that if A_μ is the vector potential of the field $F_{\mu\nu}$ satisfying the symmetry condition

$$\mathcal{L}_\xi A^\mu \equiv \xi^\alpha A^\mu{}_{;\alpha} - A^\alpha \xi^\mu{}_{;\alpha} = 0, \tag{11.2.27}$$

we can choose for Φ the quantity

$$\Phi = -A_\alpha \xi^\alpha. \tag{11.2.28}$$

Indeed, differentiating (11.2.28) and then using (11.2.27), we obtain

$$\Phi_{,\mu} = -A_{\alpha;\mu}\xi^\alpha - A_\alpha \xi^\alpha{}_{;\mu} = -A_{\alpha;\mu}\xi^\alpha + A_{\mu;\alpha}\xi^\alpha = E_\mu. \tag{11.2.29}$$

We also specify that the potential is chosen in such a manner that A_μ vanishes at infinity. If for ξ^μ we choose a vector field η^μ (11.2.9), the value Φ^H of the corresponding quantity Φ at the event horizon is called the *electric potential* of the

black hole. We can show that Φ^H is constant on the horizon. To do this we remark that the condition

$$T_{\mu\nu} l^\mu l^\nu = 0,$$

which is implied by (6.2.2) to hold on the surface of an arbitrary stationary black hole ($l^\mu = \eta^\mu|_H$), is equivalent for the electromagnetic field to the relation

$$(F_\mu{}^\alpha F_{\nu\alpha} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}) l^\mu l^\nu \equiv E^\alpha E_\alpha = 0,$$

that is,

$$\Phi_{,\mu} = E_\mu = 4\pi\sigma l_\mu,$$

on the horizon surface where the quantity σ for a Kerr–Newman black hole coincides with the 'surface charge density' introduced in Chapter 7 [cf. formula (7.3.1)]. Hence, the following equality holds for any vector χ^μ tangent to the event horizon:

$$\chi^\mu \Phi^H{}_{,\mu} = 0,$$

which means that the electric potential Φ^H at the event horizon is constant.[2]

The constancy, proved above, of Ω^H, κ, and Φ^H on the event horizon of a stationary black hole, is essential for the derivation of the so-called *mass formula.* This formula establishes a relationship of the black-hole mass observed at infinity with the geometric characteristics of its event horizon surface. Namely, Bardeen *et al.* (1973) proved that the black-hole mass M^∞ observed at infinity in stationary axisymmetric asymptotically flat spacetime can be written in the form

$$M^\infty = -\int_\Sigma (2T^\mu_\nu - T\delta^\mu_\nu)\xi^\nu_{(t)}\, d\sigma_\mu + 2\Omega^H J^H + \frac{\kappa}{4\pi} A, \tag{11.2.30}$$

where Ω^H is the angular velocity, J^H is the angular momentum, κ is the surface gravity, A is the area of the black hole, and T^μ_ν is the total energy-momentum tensor of the stationary axisymmetric distribution of matter and fields outside the black hole. The integration is carried out on a space-like asymptotically flat surface that intersects the event horizon along a two-dimensional surface $\partial\mathfrak{B}$. The surface Σ is chosen in such manner that $\xi_{(\phi)}$ is tangent to $\partial\mathfrak{B}$ and Σ is orthogonal to $\xi^\mu_{(t)}$ at the asymptotic infinity.

Formula (11.2.30) is proved as follows [Bardeen *et al.* (1973)]. For an arbitrary Killing vector field ξ^μ,

$$\xi_{\mu;\nu} = \xi_{[\mu;\nu]}, \tag{11.2.31}$$

$$\xi^{\mu;\nu}{}_{;\nu} = -R^\mu{}_\nu \xi^\nu. \tag{11.2.32}$$

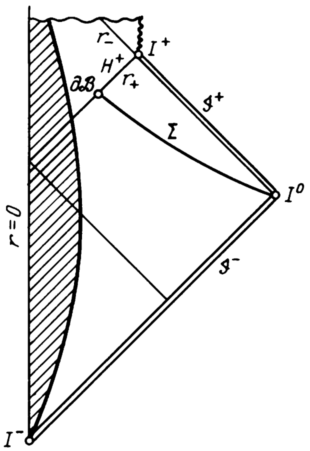


Fig. 83. Spacetime of a stationary black hole (illustration to the derivation of the mass formula).

The last of these relations can be obtained by contracting (A.15) in α and β. Choosing $\xi^{\mu}_{(t)}$ for ξ^{μ}, integrating (11.2.32) in the surface Σ, and using Stokes' theorem [see (A.33)], we find

$$\int_{\partial\Sigma} \xi^{\mu;\nu}_{(t)} \mathrm{d}\sigma_{\mu\nu} = -\int_{\Sigma} R^{\mu}{}_{\nu} \xi^{\nu}_{(t)} \mathrm{d}\sigma_{\mu}, \tag{11.2.33}$$

where $\mathrm{d}\sigma_{\mu\nu}$ and $\mathrm{d}\sigma_{\mu}$ are elements of the surface Σ and its boundary $\partial\Sigma$. If the surface Σ is chosen as described above, its boundary $\partial\Sigma$ consists of the black-hole boundary $\partial\mathcal{B}$ and a two-dimensional surface $\partial\Sigma_{\infty}$ at the spatial infinity (Figure 83).

We show below that the black-hole mass M^{∞}, measured by a distant observer in the asymptotically flat region by the effect it exerts on test particles, is given by the expression

$$M^{\infty} = \frac{1}{4\pi} \int_{\partial\Sigma_{\infty}} \xi^{\mu;\nu}_{(t)} \mathrm{d}\sigma_{\mu\nu}. \tag{11.2.34}$$

To do this, we assume that a test body is at rest far from the black hole. The four-acceleration of this body is then

$$a^{\mu} = \frac{\xi^{\nu}_{(t)} \xi^{\mu}_{(t);\nu}}{-\xi^{\alpha}_{(t)} \xi_{(t)\alpha}}. \tag{11.2.35}$$

Let Σ be a space-like surface orthogonal to the vector of the four-velocity of the body, $u^{\mu} = \xi^{\mu}/|\xi_{\alpha}\xi^{\alpha}|^{1/2}$, in the asymptotic region. The gravitational field in this region is weak so that the relation of its invariant four-dimensional characteristics to the Newtonian description is easily established. Thus, the vector $a^{\nu} \approx u^{\nu}\xi^{\mu}_{(t);\nu}$ that lies in Σ has three nonzero components. In the Newtonian theory, this three-dimensional vector characterizes the strength of the gravitational field and is related to the potential ϕ by the formula $a_i = \phi_{,i}$. By Gauss's theorem, the flux of this vector

across any closed two-dimensional surface $\partial\Sigma_\infty$ (that lies in Σ) enclosing the gravitating body equals $4\pi M^\infty$, where M^∞ is the mass of the body. Let $\tilde{n}^\mu$ be the unit vector of the outward normal to $\partial\Sigma_\infty$ that lies in Σ. Then

$$M^\infty = \frac{1}{4\pi}\int_{\partial\Sigma_\infty} \xi_{(t)}^{\mu;\nu}\tilde{n}_\mu u_\nu d^2\sigma, \tag{11.2.36}$$

where $d^2\sigma$ is an element of the area $\partial\Sigma_\infty$. Using the property (11.2.31), we can replace the expression $\tilde{n}_\mu u_\nu\, d^2\sigma$ with $d\sigma_{\mu\nu} = \tilde{n}_{[\mu}u_{\nu]}d^2\sigma$. Expression (11.2.36) then transforms to (11.2.34).

Likewise, it can be shown that the total angular momentum J^∞ of the system, measured by a distant observer (e.g., by recording the Lense–Thirring drag), is given by the following formula:[3]

$$J^\infty = -\frac{1}{8\pi}\int_{\partial\Sigma_\infty} \xi_{(\phi)}^{\mu;\nu} d\sigma_{\mu\nu}. \tag{11.2.37}$$

Using relation (11.2.33) and a similar relation for $\xi_{(\phi)}$, we can obtain, taking into account (11.2.34) and (11.2.37), that

$$M^\infty = -\int_\Sigma (2T^\mu_\nu - T^\alpha_\alpha\,\delta^\mu_\nu)\xi^\nu_{(t)}\, d\sigma_\mu + M^H, \tag{11.2.38}$$

$$J^\infty = \int_\Sigma T^\mu_\nu \xi^\nu_{(\phi)}\, d\sigma_\mu + J^H, \tag{11.2.39}$$

where the integrals in the right-hand sides describe the contributions of matter and fields outside the black hole to the total mass M^∞ and angular momentum J^∞ of the system, while

$$M^H = \frac{1}{4\pi}\int_{\partial\mathcal{B}} \xi_{(t)}^{\mu;\nu} d\sigma_{\mu\nu} \tag{11.2.40}$$

and

$$J^H = -\frac{1}{8\pi}\int_{\partial\mathcal{B}} \xi_{(\phi)}^{\mu;\nu} d\sigma_{\mu\nu} \tag{11.2.41}$$

are the contributions of the mass and angular momentum of the black hole itself to M^∞ and J^∞.[4]

The expression for M^H can be transformed as follows. Let us rephrase $\xi^\mu_{(t)}$, using (11.2.9), in terms of η^μ and $\xi^\mu_{(\phi)}$. We find

$$\int_{\partial\mathcal{B}} \xi_{(t)}^{\mu;\nu} d\sigma_{\mu\nu} = 8\pi J^H \Omega^H + \int_{\partial\mathcal{B}} l^{\mu;\nu} d\sigma_{\mu\nu}, \tag{11.2.42}$$

where $l^\mu = \eta^\mu|_H$. If the vectors m^μ and $\overline{m}^\mu$ of the complex null tetrad, introduced above, lie in the plane tangent to $\partial\mathcal{B}$, then $d\sigma_{\mu\nu} = l_{[\mu}n_{\nu]}\, dA$, where dA is an area element of the two-dimensional surface $\partial\mathcal{B}$. Using the definition (11.2.11) of the surface gravity κ and also its constancy, we can write the integral in the right-hand

side of (11.2.42) in the form $\int_{\partial\mathcal{B}} l^{\mu;\nu}\, d\sigma_{\mu\nu} = \kappa A$, where $A = \int_{\partial\mathcal{B}} dA$ is the total area of the black-hole surface. Using this equality and substituting (11.2.42) into (11.2.38), we arrive at the mass formula (11.2.30).

Recall that T^{μ}_{ν} in this formula is the total energy-momentum tensor of matter and fields outside the black hole. If the electromagnetic field is present, the tensor T^{μ}_{ν} is composed of two parts: The energy-momentum tensor of matter $T^{(m)\mu}_{\nu}$, and the energy-momentum tensor of the electromagnetic field, $T^{(em)\mu}_{\nu}$. Using the expression (A.48) for $T^{(em)\mu}_{\nu}$, we can transform formula (11.2.30) into the following form [Carter 1973a, 1979)]:

$$M^{\infty} = -\int_{\Sigma} (2T^{(m)\mu}_{\nu} - T^{(m)\alpha}_{\alpha}\,\delta^{\mu}_{\nu})\xi^{\nu}_{(t)}\, d\sigma_{\mu} + \tilde{M}^{H} + M^{(em)}_{\text{ext}}, \tag{11.2.43}$$

where $M^{(em)}_{\text{ext}}$ is the contribution to the total mass due to the presence of current j^{μ} outside the black hole,

$$M^{(em)}_{\text{ext}} = \int_{\Sigma} [-2\xi^{\nu}_{(t)}A_{\nu}j^{\mu} + j^{\nu}A_{\nu}\xi^{\mu}_{(t)}]\, d\sigma_{\mu}, \tag{11.2.44}$$

and $\tilde{M}^{H}$ is the black-hole mass including the energy of its electromagnetic field:

$$\tilde{M}^{H} = 2\Omega^{H}\tilde{J}^{H} + \Phi^{H}Q + \frac{\kappa}{8\pi}A. \tag{11.2.45}$$

Here

$$\tilde{J}^{H} = J^{H} + J^{(em)H}, \tag{11.2.46}$$

$$J^{(em)H} = \frac{1}{4\pi}\int_{\partial\mathcal{B}} \xi^{\alpha}_{(\phi)}A_{\alpha}F^{\mu\nu}\, d\sigma_{\mu\nu}, \tag{11.2.47}$$

and $Q = 1/4\pi\int_{\partial\mathcal{B}} F^{\mu\nu}\, d\sigma_{\mu\nu}$ is the electric charge of the black hole.

In the above formulas, A_{μ} denotes the vector potential of the electromagnetic field, vanishing at infinity and satisfying the relations $\mathcal{L}_{\xi_{(t)}}A_{\mu} = \mathcal{L}_{\xi_{(\phi)}}A_{\mu} = 0$.[5] If there are no matter or currents beyond the black hole, its mass M^{∞} coincides with $\tilde{M}^{H}$. Relation (11.2.45) for a lone black hole was derived by Smarr (1973a).

The integral mass formula (11.2.30) makes it possible to calculate the mass difference δM^{∞} for two static or stationary axisymmetric configurations containing a black hole. We denote the appropriate metric variation by $\delta g_{\mu\nu}$ and specify the gauge arbitrariness $\delta g_{\mu\nu} \to \delta g_{\mu\nu} + \delta\lambda_{(\mu;\nu)}$ (inherent in the choice of this quantity) in such a way that $\delta\xi^{\mu}_{(t)} = \delta\xi^{\mu}_{(\phi)} = 0$, with unchanged position of the event horizon. Then the general expression for the mass variation is written in the following form:

$$\delta M^{\infty} = \Omega^{H}\delta J^{H} + \frac{\kappa}{8\pi}\delta A - \delta\int_{\Sigma} T^{\mu}_{\nu}\xi^{\nu}_{(t)}\, d\sigma_{\mu} - \frac{1}{2}\int_{\Sigma} T^{\mu\nu}\delta g_{\mu\nu}\xi^{\alpha}_{(t)}\, d\sigma_{\alpha}. \tag{11.2.48}$$

If matter and fields are absent outside the black hole, the last two terms of this formula vanish and it becomes identical to formula (11.2.3) for the Kerr (Q=0) black hole.

We will give the explicit expression for the change δM in the total mass of the system in the case when an ideal liquid, described by the energy-momentum tensor $T_{\nu}^{(m)\mu}$, rotates outside the black hole at a local angular velocity Ω, and the system is stationary and axially symmetric both before and after its parameters undergo a change [Carter (1973a), Damour (1982)]:

$$\delta M^{\infty} = \Omega^{H}\,\delta \tilde{J}^{H} + \Phi^{H}\,\delta Q + \frac{\kappa}{8\pi}\,\delta A + \delta q, \tag{11.2.49}$$

where

$$\delta q = \int \Omega\,\delta(\mathrm{d}^{3}J^{(m)} + \xi^{\alpha}_{(\phi)}A_{\alpha}\,\mathrm{d}^{3}Q) - \int (\xi^{\alpha}_{(t)} + \Omega\xi^{\alpha}_{(\phi)})A_{\alpha}\,\delta(\mathrm{d}^{3}Q) +$$

$$+ \int \bar{T}\,\delta(\mathrm{d}^{3}S^{(m)}) + \int \bar{\mu}\,\delta(\mathrm{d}^{3}N). \tag{11.2.50}$$

Here $\mathrm{d}^3J^{(m)} = T_{\nu}^{(m)\mu}\xi^{\nu}_{(\phi)}\,\mathrm{d}\sigma_{\mu}$ is an element of local angular momentum of matter, $\mathrm{d}^3Q = j^{\mu}\,\mathrm{d}\sigma_{\mu}$ is the local charge distribution, j^{μ} is the electric current, $\bar{T} = [-(\xi_{(t)}+\Omega\xi_{(\phi)})^2]^{1/2}T$, T is the local temperature, $\mathrm{d}^3S^{(m)}$ is the local distribution of the matter entropy, $\bar{\mu} = \mu\bar{T}/T$, μ is the local chemical potential, and d^3N is the local distribution of the number of particles. The integral and differential expressions for the mass formula given in this section, are convenient for analyzing a considerable number of aspects of the processes that change the parameters of black holes. These expressions are among the main relations employed in the description of the analogues of the laws of thermodynamics in black-hole physics.

11.3. Four Laws of Black-Hole Physics

According to the thermodynamic analogy in black-hole physics the quantities

$$\theta = \frac{\hbar\kappa}{2\pi k c}, \quad S^{H} = \frac{A}{4 l_{\mathrm{Pl}}^{2}}, \quad E = Mc^{2} \tag{11.3.1}$$

(κ is the surface gravity, A is the area and $M \equiv M^{\infty}$ is the mass of the black hole) play the role of temperature, entropy, and internal energy of the black hole, respectively. Thermodynamics does not permit equilibrium when different parts of the system are at different temperatures. The existence of the state of thermodynamic equilibrium and temperature are postulated by the zeroth law of thermodynamics. A similar proposition is valid in black-hole physics.

Zeroth law of black-hole physics. The surface gravity κ of a stationary black hole is constant everywhere on the surface of the event horizon.

In the preceding section, this proposition was proved under the assumption that the energy dominance condition is satisfied.

The differential mass formula given above allows the formulation of the first law of black-hole physics.

First law of black-hole physics. When the system incorporating a black hole switches from one stationary state to another, its mass changes by

$$\delta M = \theta\, \delta S^H + \Omega^H\, \delta \tilde{J}^H + \Phi^H\, \delta Q + \delta q, \tag{11.3.2}$$

where $\delta\tilde{J}^H$ and δQ are the changes in the total angular momentum and electric charge of the black hole, respectively, and δq is the contribution to the change in the total mass due to the change in the stationary matter distribution outside the black hole [δq of the ideal liquid has the form (11.2.50)].

The following analogue of the second law of thermodynamics can be formulated in the description of *classical* processes in the field of black hole, in which Hawking's theorem holds.

Second law of black-hole physics. In any classical process, the area of a black hole, A and, hence, its entropy S^H, do not decrease:

$$\delta S^H \geq 0. \tag{11.3.3}$$

In both cases (thermodynamics and black-hole physics), the second law signals the irreversibility inherent to the system as a whole and, thus, singles out the direction of the time arrow. In thermodynamics, the law of nondecreasing entropy signifies that the part of the internal energy that cannot be transformed into work grows with time. Quite similarly, the law of nondecreasing area of a black hole signifies that the fraction of a black hole's internal energy that cannot be extracted grows with time. As in thermodynamics, the quantity S^H stems from the impossibility of extracting any information on the structure of the system (in this case, the structure of the black hole).

Quantum effects violate the condition of applicability of Hawking's theorem. Thus, the quantum evaporation reduces the area of black holes, and inequality (11.3.3) is violated. Nevertheless, it is essential that the black-hole radiation is thermal in nature and is accompanied by a rise in entropy in the surrounding space. It can be shown that the so-called *generalized entropy* S, defined as the sum of the black-hole entropy S^H and the entropy of matter outside the black hole, S^m,

$$S = S^H + S^m, \tag{11.3.4}$$

does not decrease. To show this, we note that the rate of loss (by the clock of a distant observer) of mass and entropy of the black hole via radiation of a massless spin-s field can be written in the form

$$\frac{dM}{dt} = \frac{1}{4}\sigma_s h_s \Sigma_s \theta^4, \quad \frac{dS}{dt} = \frac{1}{3}\,\sigma_s \beta_s h_s \Sigma_s \theta^3, \tag{11.3.5}$$

where h_s is the number of polarizations of the field, $\sigma_s = \pi^2/30$ for bosons and $7\pi^2/240$ for fermions, Σ_s is the effective cross section of the black hole, θ is its temperature, and β_s is a dimensionless coefficient of order unity. On the other hand, the change in the entropy S^H of a nonrotating black hole is related to the change in its mass M by the formula

$$dS^H = \theta^{-1}\, dM. \tag{11.3.6}$$

Comparing (11.3.5) and (11.3.6), we find [Zurek (1982)]

$$R \equiv \frac{dS^m}{dS^H} = \frac{4}{3}\beta_s. \tag{11.3.7}$$

Numerical results obtained using formulas (9.5.28) and (9.5.39a) [Zurek (1982) and Page (1983)] demonstrate that the coefficient β is always greater than 3/4; hence, the generalized entropy S increases when a lone black hole emits radiation. It can be shown [Zurek (1982)] that if there is a blackbody radiation at a temperature $\tilde{\theta}$ outside the black hole, the generalized entropy again increases, except in the case of $\tilde{\theta} = \theta$. In this special case, the increase in entropy in the space around the black hole due to its evaporation is exactly compensated for by a decrease in the entropy in this space, due to the accretion of thermal radiation onto the black hole.

These arguments give sufficient grounds for assuming that the following law holds:

Generalized second law of black-hole physics. In any physical process involving black holes, the generalized entropy S defined by relation (11.3.4) does not decrease.[6]

$$\delta S \geq 0. \tag{11.3.8}$$

The current status of this law in black-hole physics resembles, to some extent, the status of the second law of thermodynamics before statistical mechanics was developed. The law is supported by various gedanken experiments discussed in the literature. Nevertheless, we are still lacking a consistent derivation of the law from the basic principles of quantum mechanics and the theory of gravitation.

Let us look in some detail at one such gedanken experiment [Bekenstein (1973b, 1974)]. Imagine that radiation (or matter) possessing energy E^m and entropy S^m is placed in a box with impenetrable (mirror) walls, and let this box be slowly lowered into a nonrotating black hole. When the bottom of the box is in the immediate vicinity of the black-hole surface, the bottom lid is opened and the contents of the box fall into the black hole. The box is then hoisted up.

Let us evaluate the change in generalized entropy in this process. Let all dimensions of the box, including its height l, be small in comparison with the black-hole radius r_g. When the lower face of the box reaches the black hole, the upper face is at $r = r_g + \Delta r$, where $\Delta r \approx l^2/4r_g$. The energy transferred to the black hole (taking into account the red shift) does not exceed the amount

$$\varepsilon = \left(1 - \frac{r_g}{r}\right)^{1/2} E^m \approx \frac{l}{2r_g} E^m,$$

and entropy increases by

$$\delta S^H \lesssim \theta^{-1}\varepsilon \approx 2\pi l E^m.$$

These arguments imply that the generalized entropy is indeed nondecreasing in this process, provided the inequality $S^m \leq 2\pi l E^m$ is satisfied.[7]

If l is chosen to be sufficiently small and much smaller than the other two dimensions of the box, one may conclude, at first glance, that the generalized second law can be violated. Unruh and Wald (1982, 1983a) proved that this conclusion is wrong. Indeed, the analysis ignored the effects due to the vacuum polarization. When these effects are taken into account, the situation is changed drastically.

The box is assumed to have mirror walls. As the box slowly approaches the black hole, the walls undergo progressively higher acceleration and the accelerated motion of mirror boundaries generates energy fluxes (due to quantum effects). This behavior is well known in flat spacetime [see De Witt (1975), Birrel and Davies (1982)]. Its physical cause is clear. When a mirror boundary reflects radiation, charges and currents are induced on it. A similar phenomenon takes place in the vacuum. The charges and currents 'induced' by zero oscillations fluctuate and their expectation values are zero. If such a body begins to move with an acceleration, the 'induced' charges and currents emit radiation. When a flat mirror moves at an increasing acceleration in flat spacetime, energy fluxes in the direction of the acceleration vector are generated on both sides of the mirror.[8]

Unruh and Wald (1982, 1983a) proved that a similar effect takes place when a box with mirror walls is lowered into a black hole. As the acceleration of the bottom is always greater than that of the upper wall, slow lowering of the box produces an additional positive energy flux due to vacuum polarization by the walls; this flux points into the black hole. As a result, the energy ε transferred to the black hole and the related amount of entropy S^H, are increased and the generalized entropy does not decrease.

Note that the radiation of the mirror walls into the box reduces the energy inside. If an empty box is lowered, the energy inside it may become negative. At a point where the acceleration is a ($a \gg \kappa$), the energy density in the box is of the order $-\sigma T_a^4$, where $T_a = a/2\pi$ is the local temperature measured by an observer at rest in this point (see Section 10.3). If we now open this box, a flux of negative energy into the black hole will be formed. This will stop when the energy-momentum density of radiation in the box becomes comparable to the energy-momentum density $\langle T^\mu_\nu \rangle$ in the neighborhood of the point $[\sim\sigma(\kappa/2\pi)^4]$. If the lid of the box is then closed and the box is pulled out, the observer finds that it is filled with thermal radiation at a temperature $T_a = a/2\pi$. As a result of this cyclic process, the mass of black hole slightly decreases and the energy equal to the difference between the energy of extracted radiation and the work done in the cycle can be utilized for doing work [Unruh and Wald (1982, 1983a)].

By repeating the cycle, it is possible to continuously extract energy, even from cold (massive) black holes. Unruh and Wald (1983b) obtained a theoretical restriction on the admissible power produced in such a process: $|dE/dt| \lesssim c^5/G \approx 3.6 \times 10^{59}$ erg/s. This process also obeys the generalized second law of black-hole physics.

The fact that the generalized law includes, on an equal footing, the seemingly very different quantities, S^m (which characterizes the 'degree of chaos' in the

structure of the physical matter) and S^H (which is a geometric characteristic of the black hole), is a new indication of their profound similarity. In fact, the very possibility of this relation is rooted in Einstein's equations that relate the physical characteristics of matter with the geometric characteristics of spacetime.

The relationship of the thermal properties of black holes to the loss of information on the spacetime region inside the hole, is in accordance with the general informational approach to thermodynamics that was formulated by Szillard (1929) and later elaborated on by a number of physicists and mathematicians [see, e.g., the monographs of Brillouin (1956) and Poplavsky (1981)]. This approach essentially consists of postulating a direct relationship between the deficit of information on the physical system and its entropy.

The information on the state of collapsed matter is 'cut off' by a strong gravitational force. The black hole 'forgets' its history, memorizing only the 'macroscopic' characteristics: mass, charge, and angular momentum. Correspondingly, the entropy S^H of a black hole is a measure of the amount of information lost as a result of collapse. The number of distinct ('microscopic') states of the system whose collapse produces a black hole with prescribed parameters M, J, Q must be proportional to exp $[S(M, J, Q)/k]$ [Bekenstein (1973b, 1980), Hawking (1976a), Wald (1979b)]. The direct calculation of this number of states is a very complicated, not yet solved problem.

Other approaches to defining the space of microscopic states of a black hole are known. We will briefly outline two of them. York (1983) noticed that quantum evaporation of a black hole thermally excites its gravitational quasinormal modes. York proposed to define the entropy of the black hole as the logarithm of the number of distinct excitation states of these modes in the process of black-hole evaporation.

Zurek and Throne (1985) relate the black-hole entropy to the logarithm of the number of distinct states that can exist in the thin surface layer outside the black hole, between the event horizon and the 'stretched' horizon (see Section 7.3).

Despite a certain success of the approaches outlined above, the rigorous microscopic definition of black-hole entropy and the justification of the generalized second law remain unsolved problems in black-hole physics.

Finally, let us formulate an analogue of the third law of thermodynamics [Bardeen *et al.* (1973)].[9]

Third law of black-hole physics. The temperature of a black hole cannot be reduced to zero by a finite number of operations.

Since θ vanishes simultaneously with κ, this is only possible if a lone stationary black hole is extremal: $M^2 = a^2 + Q^2$. The impossibility of transforming a black hole into an extremal one, via physical means in a finite number of steps, is closely related to the impossibility of realizing a state with $M^2 < a^2 + Q^2$ in which a naked singularity would appear and the 'cosmic censor' principle would be violated.

An analysis of specific examples [see, e.g., Wald (1974a)] show that the closer the state of the black hole to the extremal one, the more restrictive the conditions of the possibility of making the next step.

11.4. Black Hole as a Thermodynamic System

Let us look more closely at the case of a black hole surrounded by blackbody radiation at a temperature T.[10] We have already mentioned that if this temperature coincides with that of the black hole, θ, we deal with equilibrium in which the accretion of radiation onto the black hole is compensated for by the Hawking radiation of the hole.[11] It is not difficult to verify that this equilibrium is unstable. Indeed, let a black hole absorb, in response to a random fluctuation, less energy during some time interval than it has radiated away. In this case, its mass slightly diminishes but its temperature θ increases, which results in a further increase of the radiation rate and a further reduction of the black hole mass. On the other hand, a fluctuation increasing the mass of the black hole reduces its temperature and also the rate of the Hawking radiation. In this case, the accretion of radiation onto the black hole becomes the dominating process. In other words, two situations are possible if there is a sufficient amount of radiation around the black hole: either complete evaporation of the black hole or unlimited growth of its size.[12]

This specificity in the behavior of nonrotating, noncharged black holes is directly traced to their heat capacity,

$$C=\theta\left(\frac{\partial S^H}{\partial\theta}\right)_{J=Q=0}, \tag{11.4.1}$$

being negative ($C=-8\pi M^2$). The negative heat capacity signifies that a reduction in the system's energy increases its temperature ($dE = C d\theta$). This property is characteristic for systems with long-range attractive forces and, thus, for systems with gravitational self-action. It is readily shown (using, e.g., the virial theorem) that a reduction in the dimensions of such a system, resulting in a reduction of its potential and total energies, produces a simultaneous increase in the kinetic energy of its particles (the temperature of the body).

Let us show that if a black hole is placed in a reservoir with radiation and its energy is finite, a stable equilibrium configuration is possible. Let the temperature of radiation be T; then its energy E^m and entropy S^m are

$$E^m=\sigma VT^4,\quad S^m=\frac{4}{3}\sigma VT^3, \tag{11.4.2}$$

where V is the volume of the reservoir,

$$\sigma=\frac{\pi^2}{15}\left(n_b+\frac{7}{8}n_f+\frac{1}{2}n_s\right) \tag{11.4.3}$$

n_b is the number of boson fields with nonzero spin, n_f is the number of fermion fields, and n_s is the number of scalar fields (for the sake of simplification, we treat only massless fields). The condition of stable equilibrium in a system consisting of a reservoir with radiation of a black hole inside it is that the generalized entropy is maximal,

$$S = S^H + S^m = 4\pi M^2 + \frac{4}{3}\sigma V T^3, \tag{11.4.4}$$

for a fixed value of total energy,

$$E = M + \sigma V T^4. \tag{11.4.5}$$

Using the relation $dM/dT = -4\sigma V T^3$, implied by (11.4.5), we can show that the extremum of S is achieved if $T = \theta = 1/8\pi M$ which means that the radiation and black-hole temperatures coincide. This equilibrium is stable if $d^2S/dT^2 < 0$; this is equivalent to the inequality

$$E^m < \frac{1}{4} M. \tag{11.4.6}$$

This equilibrium is stabilized by the following mechanism. Assume, as before, that a black hole absorbed more energy in a fluctuation than it has radiated away. Its temperature and, hence, its rate of radiation, are then reduced. However, the rate of accretion onto the black hole has reduced because of the diminished amount of radiation outside it. If condition (11.4.6) is satisfied, the second of these effects is more important so that the black hole returns to the original state after its mass diminishes owing to the excess of radiation over accretion. The situation with fluctuations due to the reduction of black-hole mass is analogous.

Condition (11.4.6) can be reformulated as a restriction on the volume V. We denote

$$V_{\text{cr}} = \frac{2^{20}\pi^4}{5^5\sigma} E^5. \tag{11.4.7}$$

If $V > V_{\text{cr}}$, then the most probable state is the thermal radiation without a black hole. In the case of a reversed inequality ($V < V_{\text{cr}}$), the system contains a black hole surrounded by thermal radiation at a temperature $T = \theta$ [Hawking (1976a)]. The process of black-hole formation at $V = V_{\text{cr}}$ when V decreases resembles a first-order phase transition and is similar to the process of nucleation of a liquid droplet in the course of vapor cooling.

The heat capacity of a charged rotating black hole, calculated with a formula similar to (11.4.1), is

$$C = \frac{MTS^3}{\pi J^2 + (\pi/4)Q^4 - T^2S^3}. \tag{11.4.8}$$

If we denote $J^2 = \alpha M^4$ and $Q^2 = \beta M^2$, it is readily shown that C undergoes sign reversal at the parameters α and β that satisfy the relation [Davies (1977)]

$$\alpha^2 + 6\alpha + 4\beta - 3 = 0; \tag{11.4.9}$$

at this point, C becomes infinite. Although this property of heat capacity is typical, to a certain extent, of the properties of heat capacity of ordinary matter in second-

order phase transitions, the creation of charged particles and the quantum analogue of superradiation make it difficult to treat rigorously the physical peculiarities caused by the above-described behavior of heat capacity C [Davies (1977), Hut (1977), Sokolowski and Mazur (1980)].

The thermodynamic analogy in black-hole physics, that we have discussed in this chapter, was essentially limited to *equilibrium thermodynamics* (i.e., to considering equilibrium states and various relations between the characteristics of these states). In fact, this analogy is broader. It can also be traced in *nonequilibrium* thermodynamics which describes irreversible transitions of a system between its states and the processes taking place when the system passes to the state of thermodynamic equilibrium [Damour (1979)]. A general analysis of the problems of nonequilibrium thermodynamics of black holes can be found in Sciama (1981). On the change of a black-hole entropy in nonequilibrium processes, see Hawking and Hartle (1972), Bekenstein (1974), Carter (1979).

The main attention in the description of black holes in this and earlier chapters was focused on those properties that can be studied by a distant observer. One general remark is necessary before we pass on to discussing the structure of spacetime inside the black hole. The following standpoint was basic for discussing black holes: A black hole is an object possessing strong gravitational field and whose property of paramount importance is that it absorbs everything and does not radiate anything away. The event horizon is a nonmaterial imaginary boundary that separates the external space from the region that lets noting out.

Our understanding of black holes gradually deepened as the various processes involving black holes were studied. It was found that, to a certain extent, a black hole behaves in these processes much as any other material body would, and is characterized by a set of physical properties. The surface of the black hole is characterized by a sort of tension. In the absence of external factors, a nonrotating black hole becomes spherical. An abruptly applied force causes eigenoscillations of the black hole, which are damped out with time as if friction was present (quasinormal modes Section 3.2). A black hole in external static field is deformed as an elastic body (Section 8.5). If a black hole is rotating, the angular velocity of the rotation of its surface is constant, as for solid bodies (Section 11.2). A black hole emits radiation as a heated body (Section 9.5), has a certain entropy (Section 11.3), and obeys thermodynamic laws (Section 11.3). A black hole has surface viscosity. An external body at rest produces a tide effect on the horizon of a rotating black hole, resulting in a gradual slowdown of its rotation and growth of entropy [Hawking and Hartle (1972), Hartle (1973, 1974)]. In electrodynamic processes, a black hole behaves as if the surface charges and currents satisfying the conservation law and Gauss's, Ampere's, and Ohm's laws are induced on its surface by external fields. The surface resistance is 377 ohm (Section 7.3).

This approach to a black hole as a physical body with specific surface properties, gave rise to the so-called membrane formalism already mentioned in Section 7.3. This formalism was formulated by Damour (1978, 1979, 1982) and Znajek (1978) and then developed on the basis of the '3+1' split of spacetime by Thorne and Macdonald (1982), Price and Thorne (1985), and Thorne (1986) [for a general

review, see Thorne *et al.* (1985) and the monograph by the Thorne, Price, and MacDonald (1986)].

It is worth emphasizing once again: Although this approach somewhat facilitates the analysis of a number of effects involving black holes, and makes it possible to employ habitual physical intuition, one has to remember that a black hole has no material shell and that the approach is merely a convenient method for describing a distant observer.

Chapter 12

The Internal Structure of Black Holes

12.1. Spacetime and Physical Fields Inside a Schwarzschild Black Hole

The structure of spacetime inside a nonrotating black hole has been discussed in Chapter 2. Here we will discuss the behavior of physical fields and the problems of the stability of the interior part of a Schwarzschild black hole with $r < r_g$, by analogy to what has been performed for the space outside a black hole in Chapter 3.

This problem was solved by Doroshkevich and Novikov (1978). The properties of spacetime inside a black hole are essential for the problem of gravitational collapse and the nature of singularity, even though this region is inaccessible for an observer that stays outside the black hole. The general theorems about the properties of black holes, discussed in Chapter 5, do not give a specific expression for the structure of spacetime inside the black hole. It has sometimes been conjectured that all radiation fields and all perturbations grow under the black-hole horizon and become nonlinear, and that the structure of the metric must be most peculiar.

Furthermore, we have seen in Section 6.5 that the structure of the analytic continuation of the solution for the spacetime metric inside a charged and rotating black hole is very complicated, even fantastic. Is this structure indeed realized to some extent when a black hole is actually formed?

This question will be answered in this and subsequent sections. First we analyze the propagation of physical fields inside a Schwarzschild black hole and the stability of its internal structure.

Consider a perturbation created by a test object falling into the black hole; this object is the source of the fields of interest (scalar, electromagnetic, gravitational, etc.). We wish to clarify the properties of wave fields long after the object has fallen into the black hole, that is, when the observation point tends to I^+ (see, e.g., Figure 50c) in the T_--region (also designated as region II).

This choice of asymptotic procedure signifies that if coordinates (2.2.1) are employed, then $t \to \infty$. An analysis of the 'tails' of radiation (see Section 3.4) shows that in the external region R the radiative modes fall off, for a fixed r, by a power law as $t \to \infty$. As for the horizon itself, $r = r_g$, this means that if one introduces an affine parameter V along all geodesics forming the horizon, the perturbations will also fall off by a power law as $V \to \infty$. The evolution of the

wave field in the T_--region will be determined by the equations of types (3.1.1) and (3.1.2) for $r < r_g$.

The required mathematical analysis was carried out in the already-cited paper of Doroshkevich and Novikov (1978). The following results were obtained. For scalar perturbations,

$$\Phi \approx D_1 t^{-2(l+1)} + D_2 t^{-(2l+3)} \ln r, \tag{12.1.1}$$

where D_1 and D_2 are constants.

For perturbations described by fields with $s \neq 0$ (including metric perturbations), the main term of the r-dependent component has the following form for radiative multipoles $l \geq s$:

$$\Phi_1 \approx D_3 t^{-(2l+3)} r^{-n} \tag{12.1.2}$$

(D_3 and n are constants).

Hence, if r is fixed and $t \to \infty$, radiative modes of perturbations due to external sources are damped out and the spacetime tends to a 'stationary' state described by the Schwarzschild solution. The situation is similar to that discussed in Chapter 3 for the exterior region of the black hole. However, important differences manifest themselves at $r < r_g$.

First, the coordinate r at $r < r_g$ plays the role of the time coordinate, and t acts as the spatial coordinate. It would be more correct, therefore, to refer to the tendency to a state that depends only on r, not to a tendency to a stationary state.

The second and more important difference, is that perturbations grows infinitely as the point tends to singularity at a fixed t. The general solution close to the singularity, ignoring the quantum effects, was constructed by Belinsky, Lifshitz, and Khalatnikov (1970). The method of small perturbations does not work near the singularity. The boundary of the region where perturbations cease to be small is given by the expression (for $s \neq 0$):

$$r^n \approx D_3 t^{-(2l+3)}. \tag{12.1.3}$$

As t increases, this region contracts to the singularity. The Schwarzschild solution is stable with respect to small perturbations everywhere inside the black hole, with the exception of this region continuously contracting as $t \to \infty$, and all radiative modes are damped out by a power law as t increases.

Radiation from elementary perturbations produced in the region of $r < r_g$ propagates only to a finite (small) region inside the black hole, because the emitted signals 'ram' into the singularity. These perturbations do not affect in any way the properties of the black hole interior at $t \to \infty$.

The most important conclusion that can be drawn from this discussion is that the Schwarzschild solution is as stable in the T_- -region as it is in the R region.

Several words are necessary on nonradiative perturbation multipoles connected with the particles falling into the black hole or with the collapsing body that produces the black hole. This multipole is $l = 0$ for electromagnetic perturbations

(the Coulomb field of the falling charge), and $l = 0$ (the field of additional mass) and $l = 1$ (the field of angular momentum) for gravitational perturbations. These multipoles do not damp out as $t \to \infty$. They increase in the familiar manner as $r \to 0$, and reconstruct the metric near $r = 0$, because this situation corresponds to the transition to the Reissner–Nordström metric if electric charge is added and to the Kerr metric if the angular momentum is added. We will later discuss these metrics in relation to the internal structure of black holes. Here, we only emphasize that if the corrections to the metric and the perturbation fields become significant sufficiently close to the singularity $r = 0$, they have no immediate physical meaning. The point is that the quantum processes become important in the vicinity of the singularity where the spacetime curvature exceeds the Planck value (i.e., becomes greater than $1/l_{\rm Pl}^2$); for instance, for the quadratic invariant of curvature, we have

$$R_{iklm}R^{iklm} = \frac{12r_g^2}{r^6} > \frac{1}{l_{\rm Pl}^4}. \tag{12.1.4}$$

The entire region defined by (12.1.4) may be regarded as a 'singularity' from the standpoint of the classical theory.

The last point is the case of nonradiative multipoles of physical fields, connected with external sources. If the sources are stationary, that is, the characteristic time of field variation $t \gg r_g/c$, then, as shown in Section 4.3, the fields of these sources with $k = 0$ unobstructedly penetrate into the black hole through the potential barrier. We assume these fields to be weak on r_g. The fields are independent of the coordinate t both inside and outside the black hole.

Typical examples to be discussed here are the effect of an external stationary quadrupole gravitational field and that of a stationary magnetic field, homogeneous far from the black hole, on the internal structure of this black hole.

The exact solution describing a black hole in an external quadrupole field [Doroshkevich *et al.* (1965)] was given in Section 8.5.

Expression (8.5.34) shows that if the parameter q of the quadrupole moment is sufficiently small so that the corrections to the metric at $r = r_g$ are small, then the corrections remain small everywhere for $r < r_g$ up to the singularity at $r = 0$. No instability is produced thereby inside the black hole.

Now we will consider a magnetic field that is homogeneous at the spatial infinity. Let us assume this field to be weak. Then we find that the nonzero components of the electromagnetic field tensor are [Ginzburg and Ozernoi (1964)]

$$F^{\theta\phi} = \frac{B_0 \cot\!an\, \theta}{r^2}, \quad F^{\phi r} = -\frac{B_0}{r}\left(1 - \frac{r_g}{r}\right), \tag{12.1.5}$$

where B_0 is the magnetic field strength at infinity. The component $F^{\phi r}$ describes the electric field inside the black hole. As $r \to 0$, the components of the electromagnetic field tensor grow infinitely. This does not however, restructure the metric, because the components of the energy-momentum tensor constructed of components

(12.1.5) grow at a slower rate than the corresponding expressions describing spacetime curvature.

Let us again point out that a physical 'singularity' for which quantum gravity processes are essential, is created in the vicinity of the true mathematical singularity $r = 0$. Moreover, quantum phenomena in the presence of the fields other than the gravitational (e.g., in a charged black hole) can also arise farther away from the singularity (where curvature is not large). In some cases, these phenomena greatly influence the metric. They are to be considered in the subsequent sections.

Finally, the Hawking evaporation of black holes affects the metric on a very large time scale.

12.2. Instability of Cauchy Horizons Inside a Charged Spherical Black Hole

Consider the behavior of small perturbations of the gravitational and electromagnetic fields inside a charged spherically symmetric black hole.

A qualitatively new factor appears here in the comparison with the case of the Schwarzschild black hole: the presence of the Cauchy horizons (see Section 6.5). Figure 84 shows a fragment of the Penrose diagram with the internal part (region II) of the charged black hole and the external space. If the charged black hole is a result of collapse of the charged body from space I, then the other external space (I' in Figure 67) is absent, as it is in the case of the collapse of a noncharged spherical body that transforms into a Schwarzschild black hole (see Section 2.7). This is why

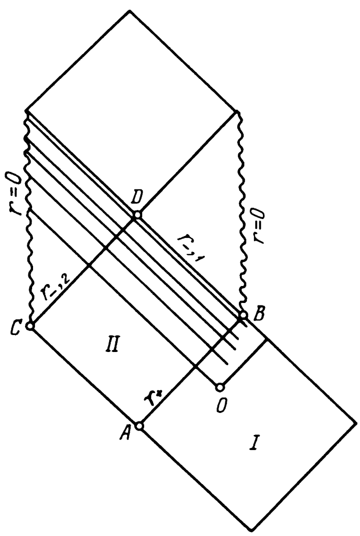


Fig. 84. Part of the Penrose diagram for a charged black hole. The propagation of radial rays is shown immediately after the flash (0) and after the scattering on the spacetime curvature.

region I' is not shown in Figure 84. There are serious reasons (see Section 6.5) for expecting that small perturbations can grow infinitely in the neighborhood of $r_{-,1}$ [Penrose (1968)].

Indeed, let us consider a small perturbation of the gravitational and (or) electromagnetic field outside the black hole in region I. We have already shown in Section 3.4 and 3.7 that the 'tails' of radiation induced by perturbations damp out following a power law dependence at r = const, as a result of scattering on the spacetime curvature. This decreasing radiation flux crosses the event horizon r_+ and concentrates along the horizon $r_{-,1}$ (see Figure 84). An observer travelling along a time-like world line and crossing the horizon $r_{-,1}$ will receive this radiation close to $r_{-,1}$ in a finite proper time (this radiation sinks into the black hole after an infinite time of the external observer). When the observer approaches $r_{-,1}$, the radiation he receives has infinite blue shift. It is natural to expect that such concentration of energy restructures the spacetime and produces a true spacetime singularity instead of $r_{-,1}$. At the same time, no concentration of energy appears along the horizon $r_{-,2}$ (except at the point D), so that we cannot expect that perturbations appearing in region I 'create' a singularity at $r_{-,2}$.

We will see that a mathematical analysis of the evolution of small perturbations confirms these intuitive arguments.

This problem was analyzed by McNamara (1978a, b), Gürsel *et al.* (1979a, b), Matzner *et al.* (1979), and Chandrasekhar and Hartle (1982). Here, we follow the last of these papers.

The metric of a charged black hole is

$$ds^2 = \frac{-(r-r_+)(r-r_-)}{r^2}dt^2 + \frac{r^2}{(r-r_+)(r-r_-)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \qquad (12.2.1)$$

We are interested in the spacetime region $r_- < r < r_+$. Chandrasekhar (1979b, 1983) and Chandrasekhar and Xanthopoulos (1979) proved that the perturbations of the gravitational and electromagnetic fields of a charged Reissner–Nordström black hole can be analyzed in terms of normal modes, with time dependence in the form $e^{i\sigma\tau}$, and angular momentum described by the appropriate Legendre functions with fixed l and m. Perturbation are subsumed under two classes: axial [superscript (+)] and polar [superscript (–)]. A perturbations of each class can be expressed in terms of a pair of gauge-invariant (in the sense outlined in Section 3.1) scalar functions $Z_i^{(\pm)}(r)$ ($i = 1, 2$) that satisfy the equation

$$\frac{d^2 Z_i^{(\pm)}}{dr_*^2} + \sigma^2 Z_i^{(\pm)} = V_i^{(\pm)} Z_i^{(\pm)}, \qquad (12.2.2)$$

where

$$r_* = r + \frac{1}{2K_+}\ln|r_+ - r| - \frac{1}{2K_-}\ln|r_- - r|, \qquad (12.2.3)$$

$$K_+ = \frac{r_+ - r_-}{2r_+^2}, \quad K_- = \frac{r_+ - r_-}{2r_-^2}, \tag{12.2.4}$$

$$V_i^{(-)} = \frac{\Delta}{r^5}\left[(\mu^2 + 2)r - q_j + \frac{4Q^2}{r}\right], \tag{12.2.5}$$

$$V_i^{(+)} = V_i^{(-)} + 2q_j \frac{d}{dr_*} \frac{\Delta}{r^2(\mu^2 r + q_j)}, \tag{12.2.6}$$

$$\Delta = (r - r_+)(r - r_-) = r^2 - 2Mr + Q^2, \quad \mu^2 = (l - 1)(l + 2), \tag{12.2.7}$$

$$q_1 = 3M + \sqrt{(9M^2 + 4Q^2\mu^2)}, \quad q_2 = 3M - \sqrt{(9M^2 + 4Q^2\mu^2)} \tag{12.2.8}$$

[in formulas (12.2.5), (12.2.6), $i, j = 1, 2; i \neq j$)]. It is found that $Z_i^{(+)}$ is related to $Z_i^{(-)}$ by a simple algebraic transformation.

We have already encountered equations of type (12.2.2) in the problem of behavior of physical fields outside a black hole (see Chapter 3). We are interested in the solution of these equations which describes the transmission and reflection of the wave incident on $V_i^{(\pm)}$. (In the present case, $V_i^{(\pm)}$ is a potential well, not a barrier as was the case for the exterior space of the black hole, but the situation remains qualitatively unchanged.) It is found that the reflection and transmission coefficients for $Z_i^{(+)}$ are expressed simply in terms of the corresponding coefficients for $Z_i^{(-)}$.

Let us analyze the behavior of wave perturbations that arrive at region II across the horizon r_+ from region I (see Figure 84). To do this, we consider the dispersion of a wave of unit amplitude on $r = r_+$ ($r_* \to -\infty$). The solution of equations (12.2.2) must satisfy the following boundary conditions (the upper and lower indices are dropped because the analysis holds for all their values):

$$Z(r_*, \sigma) \to A(\sigma)\, e^{-i\sigma r_*} + B(\sigma)\, e^{+i\sigma r_*} (r_* \to +\infty,\ r \to r_-), \tag{12.2.9}$$

$$Z(r_*, \sigma) \to e^{-i\sigma r_*} (r_* \to -\infty,\ r \to r_+). \tag{12.2.10}$$

The coefficients $A(\sigma)$ and $B(\sigma)$ describing the transmission and the reflection of the wave at $V(r_*)$, can be found, in principle, by standard techniques, provided the form of $V(r_*)$ is known [see (12.2.5), (12.2.6)]. The analysis is facilitated by introducing the null coordinates

$$\tilde{u} = r_* + t, \quad \tilde{v} = r_* - t. \tag{12.2.11}$$

The lines of constant $\tilde{u}$ in Figure 84 (not shown in order, so as to avoid crowding the Figure) would appear as segments of straight line parallel to r_+. The lines of constant $\tilde{v}$ (not traced in the figure either) would appear as segments of straight

lines parallel to $r_{-,1}$. The horizon r_+ corresponds to $\tilde{u} = -\infty$, the horizon $r_{-,1}$ to $\tilde{v} = \infty$, and the horizon $r_{-,2}$ to $\tilde{u}\tilde{v} = +\infty$. Boundary conditions (12.2.9), (12.2.10) are rewritten in the form

$$Z(r_*, t) \to e^{-i\sigma\tilde{v}} + [A(\sigma) - 1]\, e^{-i\sigma\tilde{v}} +$$

$$+B(\sigma)\, e^{+i\sigma\tilde{u}} \quad (r_* \to \infty, r \to r_-). \tag{12.2.12}$$

$$Z(r_*, t) \to e^{-i\sigma\tilde{v}} \quad (r_* \to -\infty, r \to r_+). \tag{12.2.13}$$

Consider a perturbation $Z_{\text{pert}}(\tilde{v})$ intersecting the horizon r_+, that is, given for $\tilde{u} \to -\infty$. Its Fourier transform is

$$\tilde{Z}(\sigma) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z_{\text{pert}}(\tilde{v})\, e^{i\sigma\tilde{v}}\, d\tilde{v}. \tag{12.2.14}$$

Having undergone dispersion in region II, the perturbation reaches the horizon r_-. Its amplitude there can be written in the form [see (12.2.12)]

$$Z_{\text{scat}}(r_*, t) \to X(\tilde{v}) + Y(\tilde{u}) \quad (r_* \to \infty), \tag{12.2.15}$$

where

$$X(\tilde{v}) = \int_{-\infty}^{\infty} \tilde{Z}(\sigma)[A(\sigma) - 1]\, e^{-i\sigma\tilde{v}}\, d\sigma, \tag{12.2.16}$$

$$Y(\tilde{u}) = \int_{-\infty}^{\infty} \tilde{Z}(\sigma)B(\sigma)\, e^{+i\sigma\tilde{u}}\, d\sigma. \tag{12.2.17}$$

We are interested in finding the radiation flux received by the observer crossing the horizon r_-. This flux is proportional to the squared amplitude

$$F = u^\alpha Z_{,\alpha}\big|_{r \to r_-}, \tag{12.2.18}$$

where u^α is the four-dimensional velocity of the observer. The horizon r_- is either stable or unstable (in the linear approximation) with respect to small perturbations depending on whether the flux is finite or infinite.

Matzner *et al.* (1979) and Chandrasekhar (1983) have shown that the quantity F at the Cauchy horizon is of the form

on $r_{-,1}$:

$$F_{r_{-,1}} = \frac{2r_-^2}{r_+ - r_-}\, |E| \lim_{\tilde{v} \to \infty} e^{\kappa_- \tilde{v}} X_{,-\tilde{v}}\,; \tag{12.2.19}$$

on $r_{-,2}$:

$$F_{r_{-,2}} = \frac{2r_-^2}{r_+ - r_-} E \lim_{\tilde{u}\to\infty} e^{K_- \tilde{u}} Y_{,\tilde{u}}\,, \tag{12.2.20}$$

where E is the constant time component of the covariant four-velocity u_α of the observer. Expressions (12.2.19) and (12.2.20) are divergent or finite, depending on the behavior

$$e^{K_- \tilde{u}} X_{,-\tilde{v}}|_{\tilde{v}\to\infty} \quad \text{and} \quad e^{K_- \tilde{u}} Y_{,\tilde{u}}|_{\tilde{u}\to-\infty}.$$

We assume that the form taken by perturbing radiation $Z_{\text{pert}}(v)$ crossing the horizon r_+ satisfies the following conditions:

$$Z_{\text{pert}}(\tilde{v}) = 0 \;\text{ for }\; \tilde{v} < \tilde{v}_0 \;\text{ and }\; Z_{\text{pert}}(\tilde{v}) \;\text{ for }\; \tilde{v}\to\infty \text{ tends to zero at least as } \tilde{v}^{-1}.$$

It is to these conditions that any real radiation must satisfy, for example, the radiation from an object falling into the black hole or from an elementary perturbation that has occurred in region I. Indeed, the second condition must hold for $\tilde{v}\to\infty$ at the horizon of a noncharged black hole, in accordance with the asymptotic behavior of perturbing radiation analyzed in the preceding section. A charge $Q < M$ does not change anything in this respect [see, e.g., Bičak (1972)]. The power-law decay of radiation 'tails' from perturbations is typical for practically any perturbation.

The first condition is definitely met if $\tilde{v}_0$ is interpreted as an affine parameter corresponding to the moment when the horizon is intersected by the first ray reaching it from the perturbation.[1]

Chandrasekhar and Hartle (1982) proved that the quantity $F_{r_{-,2}}$ of (12.2.20) remains finite for the solutions of Equation (12.2.2) with arbitrary superscripts ($\pm$), that is, the horizon $r_{-,2}$ is stable with respect to small perturbations in region I. To the contrary, the quantity $F_{r_{-,1}}$ diverges as $\tilde{v} \to \infty$ at least as $\exp(K_- - K_+)\,\tilde{v}$ or even faster (the rate of divergence depends on the type of perturbation). This means that as $r_{-,1}$ is approached, the observer receives radiation flux of infinite density.

The mathematical analysis presented above has totally confirmed Penrose's intuitive arguments given at the beginning of the section. Note that the dispersion of wave due to any perturbation generated in region II cannot produce energy concentration along $r_{-,1}$ and, hence, cannot result in the instability of Cauchy horizons. Infinite concentration of energy close to $r_{-,1}$ due to perturbations in region I, will inevitably affect the metric, restructuring the spacetime. The method of small perturbations is thus invalid close to $r_{-,1}$. It can only be conjectured that the true singularity of the spacetime will be formed to replace the Cauchy horizon.

Another remark needs to be made here. Let the sources of a constant external field (electromagnetic, gravitational, or some other) be present outside a black hole. If the black hole is charged, these fields penetrate through r_+ into the interior region, as happens in the case of a noncharged Schwarzschild black hole (see Section 12.1). If the fields outside the black hole are weak and do not affect the metric, they remain

weak inside the black hole as well. Thus, they are weak on r_- and do not produce any instability [for a justification of this statement, see Gürsel *et al.* (1979b)].

12.3. Instability of Cauchy Horizons with Respect to Quantum Electrodynamic Processes

The preceding section discussed the instability of the Cauchy horizon under small external perturbations. However, the method of small perturbations employed cannot answer the question about how the metric is changed by the growing small perturbations and whether the true spacetime singularity will thereby be produced.

The present section is devoted to quantum electrodynamic processes generated inside a charged nonrotating black hole; they were ignored in our analysis of the internal structure. It will be shown that these processes which create electron-positron pairs lead to the instability of the Cauchy horizon and restructure the spacetime. A self-considered solution is successfully constructed thereby, taking into account the effect of the new particles on the electromagnetic field and the metric. Besides, it is shown in the framework of this solution how the metric is modified and it is proved that the true spacetime singularity is indeed created instead of the Cauchy horizon. This problem was solved by Novikov and Starobinsky (1980c) whose paper we follow here [see also Berezin (1980)].

Consider the restrictions imposed on the physical conditions inside black holes with different M and Q (Figure 85). First, a black hole is formed only if $Q < \sqrt{G}\,M$ [or $Q/e < 5 \times 10^5\, M$ (grams) where e is the electron charge], that is, if the parameters lie below line 1 in Figure 85. If the charge of a black hole is sufficiently large, electron-positron pairs are created near it [Markov and Frolov (1970), Gibbons (1975), Damour and Ruffini (1975)]. One particle escapes to infinity, the other (its charge sign is apposite to that of a black hole) is absorbed by the black hole and diminishes its charge.[2] It can be shown that in a very short time r_+/c this charge is reduced to the value

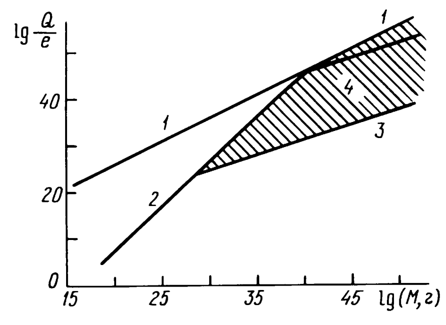


Fig. 85. Various ranges of values of the black-hole charge Q and mass M (on the boundaries of these ranges, see text).

$$Q_2 = \frac{\pi m^2 r_+^2 c^3}{e\hbar} (\ln A - \ln\ln A)^{-1}, \tag{12.3.1}$$

where $A = e^2(2\pi\hbar c)^{-1} (r_+/\lambda)^2$, e is the electron charge, and m is its mass. Subsequently, the black-hole charge remains very nearly constant.

Numeral 2 in Figure marks the line corresponding to Equation (12.3.1). The region of allowed values of black-hole parameters lies below lines 1 and 2.

Note that if the black-hole charge is sufficiently small, the Cauchy horizon lies so close to the true singularity that the spacetime curvature here exceeds the critical value at which quantum gravity effects are appreciable. This entire region must be regarded as singular form the physical standpoint. A nonsingular Cauchy horizon exists only if it lies outside this region. The curvature invariant $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ is of dimension $(\text{length})^{-4}$. The boundary of the singular region is found from the condition $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = 1/l_{\rm Pl}^4$. For the Reissner–Nordström metric, the condition of r_- belonging to the boundary of the singular region is the expression (for $r_- \ll r_+$)

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}\big|_{r=r_-} \approx \frac{12 r_+^2}{r_-^6} = \frac{1}{l_{\rm Pl}^4} \tag{12.3.2}$$

or

$$\frac{Q_3}{c} \approx G^{1/3}(\hbar c)^{1/6}\frac{M^{2/3}}{c} \approx 3 \times 10^4 M^{2/3}(\text{g}) \tag{12.3.3}$$

(line 3 in Figure 85). On this boundary, $r_- \gg l_{\rm Pl}$. If the parameters of the black hole lie to the right and below line 3, no nonsingular Cauchy horizon exists. (In natural astrophysical conditions, the relations given at the beginning of Section 4.8 are satisfied; hence, black holes with mass $M < 10^{60}$ definitely have $Q < Q_3$ and, thus, cannot have a nonsingular Cauchy horizon.)

The last step is an analysis of the region hatched in Figure 85. Quantum-electrodynamics processes inside a black hole are important if the parameters fall into this region.

Let us introduce the following reference frame, similar to (2.4.9), in region II of the Reissner–Nordström metric;

$$ds^2 = -d\tau^2 + a^2(r(\tau))\, dx^2 + r^2(\tau)(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{12.3.4}$$

where

$$\tau = -\int_{r_-}^{r} dr \frac{r}{[(r_+ - r)(r - r_-)]^{1/2}}, \tag{12.3.5}$$

$$a(r) = \frac{[(r_+ - r)(r - r_-)]^{1/2}}{r}. \tag{12.3.6}$$

The former coordinate t (for $r > r_+$) is now space-like. We denote it by $x (x \equiv t)$. The functions $a = a(\tau)$ and $r = r(\tau)$ are given by formulas (12.3.5) and (12.3.6). The reference frame with metric (12.3.4) has a homogeneous (but anisotropic) three-dimensional space, which makes it especially convenient for calculations. The three-dimensional space is infinite in coordinate x ($-\infty < x < +\infty$). This reference frame lives for a finite proper time. Its evolution begins at a moment corresponding to the value $r = r_+$ when the expansion, described by $a(\tau)$, begins in the direction x. At the start of the expansion, $a = 0$. Here (at $r = r_+$), we find a false (coordinate) singularity. Cross-sections of our reference frame in transverse directions are spheres of radius $r(\tau)$. As τ increases, the spheres monotone contract from the initial values $r = r_+$. At later τ, the expansion in the x coordinate is replaced by contraction, so that at $r = r_-$ the quantity a again vanishes, that is, we again come to a coordinate singularity. The radius of the spheres at this moment is $r = r_-$.

The electromagnetic field in the outlined reference frame is purely electric (the nonzero component of the electromagnetic tensor is $F_{\tau x}$), it is directed along x, and is independent of x. As τ increases, the strength of this field increases in inverse proportion to r^2. If this field is sufficiently strong, it creates electron-positron pairs. Particles created in region II cannot escape from the black hole because its boundary r_+ lies in the absolute past. As a result, they cannot affect the properties of the external space I but may greatly modify the situation inside the black hole. We will show that the gravitation of the created particles changes the metric in region II, whereby producing the true singularity instead of the Cauchy horizon.

Let us look into details of this process. In Figure 85, we single out the values of the parameters in region II ($r_- < r < r_+$) where the electric field reaches the value $E_{cr} = \pi m^2 c^3 / e\hbar$ at which electron-positron pairs are intensively produced. The condition under which the electric field $E = Q/r^2$ assumes the critical value at the horizon r_- is $Q/r_-^2 = E_{cr}$. This relation can be rewritten in the form (assuming $Q < \sqrt{G}\,M$), $c = 1$)

$$\frac{Q}{e} = \left(\frac{4}{\pi}\frac{\hbar M^2}{m^2 e^2}\right)^{1/3} \approx 6\times 10^{18}\, M^{2/3}(\mathrm{g}) \tag{12.3.7}$$

(line 4 in Figure 85). The condition $E > E_{cr}$ in region II is satisfied for parameters lying between lines 2, 3, and 4. In the region that we consider now, $Q \ll \sqrt{G}\,M$, that is, $r_- \ll r_+$. In this case ($r_- \ll r \ll r_+$) the evolution of system (12.3.4) follows the law

$$r(\tau) = \left(\frac{3}{2}\right)^{2/3} r_+^{1/3} |\tau|^{2/3}, \quad a(\tau) = \left(\frac{r_+}{r}\right)^{1/2} = \left(\frac{2}{3}\right)^{1/3} \left|\frac{r_+}{\tau}\right|^{1/3}. \tag{12.3.8}$$

At this stage, the electric field cannot yet influence the evolution of the metric (it is the same as for $Q = 0$, $r_- = 0$). If solution (12.3.8) continued to $\tau = 0$ (as in the case of $Q \equiv 0$, $E \equiv 0$), it would result in the true singularity $\tau = 0$. In the course of contraction of the system, the electric field varies as $E = Q/r^2$ and, in our case,

reaches the level E_{cr} at the stage (12.3.8), after which $e^{\pm}$ pairs are intensively created and accelerated by the electric field, so that the electric current is produced. This current induces an important back-reaction on the electric field. Without this reaction the field would become higher than E_{cr} and would finally change the form of (12.3.8) at r comparable with r_- and produce a false singularity at the Cauchy horizon.

Novikov and Starobinsky (1980c) showed that the field E in region II cannot be considerably greater than E_{cr}, otherwise the current due to the pair production would bring the field down to E_{cr} in a short time $\tau_0 \approx 10^{-18} - 10^{-20}$ s. It is of interest to note that the field is reduced, not in a monotone manner, but through oscillations in which the sign of the field and the current direction are reversed. The relaxation of oscillations proceeds via radiative loss by particles and plasma instabilities.

The electric field thus cannot be substantially greater than E_{cr} as long as the characteristic time of field enhancement during the contraction of the system (we ignore the effect of particles being created) is greater than the relaxation time ($|\tau| > \tau_0$). In view if this, the metric is not affected by the electric field. Neither is it influenced by the gravitation of the particles created. Indeed, the energy density of the new particles transported by the field along the x axis increases in proportion to $a^{-2}r^{-2} \propto |\tau|^{-2/3}$, and after relaxation, in proportion to $(ar^2)^{-4/3} \propto |\tau|^{-4/3}$. At the same time, the spacetime curvature increases faster than energy density ($\propto 1/|\tau|^2$).

When the stage $|\tau| < \tau_0$ is reached in the course of evolution, the creation of particles and their motion cease to significantly affect the electric field. As a result, the electric field at $|\tau| < \tau_0$ grows in proportion to r^{-2}:

$$E \approx E_{cr}(\tau_0/|\tau|)^{4/3}. \qquad (12.3.9)$$

The field begins influencing the metric when the quantity $8\pi G T_0^0$ (T_0^0 being a component of the energy-momentum tensor of the electric field) becomes equal, by an order of magnitude, to the leading terms in Einstein's equation describing the spacetime curvature. These are terms of order $1/\tau^2$. Having equated

$$8\pi G T_0^0 = GE^2 = GE_{cr}^2(\tau_0/|\tau|)^{8/3} \text{ to } 1/\tau^2,$$

we find

$$|\tau| = G^{3/2} E_{cr}^3 \tau_0^4. \qquad (12.3.10)$$

Using the numerical value E_{cr} and $\tau_0 \approx 10^{-18} - 10^{-20}$ s, we obtain $|\tau| \approx 10^{-46}$–10^{-54} s $\ll \tau_{Pl}$. Recall that the true physical singularity occurs at $|\tau| \approx \tau_{Pl}$. Hence, the quantum effects change the electric field and, consequently, the metric, restructuring it in such a way that it is now described by expressions (12.3.4) and (12.3.8) up to $|\tau| \approx \tau_{Pl}$ Furthermore, the true singularity appears instead of the Cauchy horizon.

The point to be emphasized is that we have constructed a self-consistent solution without resorting to the method of small perturbations as we did in Section 12.1 and 12.2. The solution obtained gives an exact description (of course, as long as the

theory is valid) of the formation of the true singularity, what is left is the region of parameters between lines 4 and 1 in Figure 85.

Here, the electric field in region II is less than E_{cr} so that E is not appreciably affected by pair creation. Nevertheless, pairs get produced, and even a small number of them result in the formation of a true singularity.

Qualitatively, this is easy to understand. The electric field accelerates the particles created (carrying $e^+$and e^- in opposite directions) and, thus, produces an electric current. The net three-dimensional momentum of the beams is zero. Macroscopically, one can assume that the plasma as a whole is at rest in reference frame (12.3.4), although its pressure is extremely anisotropic (unless the relaxation of the fluxes has taken place). The world lines of elementary volumes of plasma coincide with the world lines of reference frame (12.3.4) until we take into account the back reaction of the new particles on the metric. We find that, in this case, the world lines concentrated along the Cauchy horizons, like the radiation due to perturbations that concentrated in region I in Figure 84.

Novikov and Starobinsky (1980c) constructed a self-consistent solution describing this situation. The gravitation due to the newborn plasma starts affecting the solution when r approaches r_-. Solution (12.3.8) ceases to hold when $r \approx r_-$. The reconstructed solution is of the form

$$a \sim |\tau|, \quad r \sim \sqrt{\ln |\tau|}\,, \qquad \varepsilon_e \sim \frac{1}{\tau^2 \ln |\tau|} \gg \varepsilon_{(em)}, \tag{12.3.11}$$

where ε_e is the energy density of pairs created, and $\varepsilon_{(em)}$ is the energy density of the electric field. Solution (12.3.11) continues until the true singularity is produced. No Cauchy horizon is thus formed in this case. What is formed is a true singularity, with the self-consistent solution, describing the process, not being obtained by the method of small perturbations.

In conclusion, we compare the instabilities of Cauchy horizon due to quantum effects with the classical instabilities due to external perturbations that we treated in the preceding section. Which instabilities are stronger? Obviously, a quantum instability is stronger when pairs are created intensively (for $E \approx E_{cr}$) far from r_-, because it restructures the solution also far from r_-. If $E \ll E_{cr}$ both instabilities manifest themselves only very close to r_-, and the classical instability may prove stronger.

12.4. Instability of Cauchy Horizons Inside a Rotating Black Hole. General Remarks

Novikov and Starobinsky (1980a, b) presented the results showing that, in a rotating black hole, external perturbations result in the instability of Cauchy horizons and replace them with a true singularity, by analogy to what happens in the case of a charged black hole (see Section 12.2). Moreover, quantum effects are likely to arise inside a rotating black hole and also replace Cauchy horizons with a true singularity (Section 12.3).

The arising singularity is, in all cases, space-like, as it is in the Schwarzschild solution. It bounds region II from above, instead of Cauchy horizons.

To summarize: When a real black hole is formed, the structure of the part inside the event horizon is similar to that of the Schwarzschild black hole. The solution cannot be continued beyond the true singularity. As a result, anything similar to the analytic continuation of the solution, shown in Figure 67, would be impossible. As for the processes inside a true singularity, they can be treated only in the framework of a unified quantum theory incorporating gravitation.

Chapter 13

Primordial Black Holes and the Ultimate Fate of Black and White Holes

13.1. Primordial Black Holes

We already know that a black hole is created when a mass contracts to a size less than its gravitational radius (Section 2.6). Stars of sufficiently large mass must convert into black holes at the end of their evolution [Zel'dovich and Novikov (1967, 1971)]. The smaller the mass of matter, the greater the density to which it must be compressed in order to create a black hole. Powerful pressure develops at high densities, counteracting the compression. As a result, black holes of mass $M \ll M_{\odot}$ cannot form in the contemporary Universe. However, the density of matter at the beginning of expansion in the Universe was enormously high.

Zel'dovich and Novikov (1966, 1967), and then Hawking (1971a) hypothesized that black holes could have been produced at the early stages of the cosmological expansion of the Universe. Such black holes are known as *primordial*. Very special conditions are needed for primordial black holes to form. Lifshitz (1946) proved that small perturbations in a homogeneous isotopic hot Universe (with the equation of state of the matter $p = \varepsilon/3$) cannot produce appreciable inhomogeneities. A hot universe is stable under small perturbations [see Bisnovaty-Kogan *et al.* (1980)]. Large deviations from homogeneity must exist from the very beginning in the metric describing the Universe (i.e., the gravitational field had to be strongly inhomogeneous) even though the spatial distribution of matter density close to the beginning of the cosmological expansion was very uniform. When the quantity $l = ct$, where t is the time elapsed since the Big Bang, grows in the course of expansion to a value in the order of the linear size of an inhomogeneity of the metric, the possibility appears for the formation of a black hole of the mass contained by the time t in the volume l^3. The formation of black holes with masses substantially smaller then stellar masses was thus possible, provided that such holes were created at a sufficiently early stage (see below).

Primordial black holes are of special interest because Hawking's quantum evaporation is important for small-mass black holes, while only primordial black holes can have such masses. (Note that quantum evaporation of massive and even supermassive black holes may be essential for the distant future of the Universe.)

Here we leave aside the astrophysical aspects of the problem [see Novikov *et al.* (1979)] and discuss only selected problems of principal importance connected with

the possibility of primordial black-hole formation in the early Universe. [1] First of all, the following two questions arise:

(1) How large must the deviations from the metric of a homogeneous isotropic model of the Universe be for black holes to be born?

(2) What is the behavior of the accretion of the surrounding hot matter to the created hole and how does the accretion change the mass of the hole?

The second question arises because of a remark made in the pioneer paper of Zel'dovich and Novikov: If a stationary flux of gas to the black hole builds up, the black-hole mass grows at a catastrophically fast rate. But if such stationary accretion does not build up immediately after the black hole is formed, accretion is quite negligible at later stages because the density of the surrounding gas in the expanding Universe falls off very rapidly.

Both questions can be answered via numerical modeling. The required computation for the case of spherical symmetry were carried out by Nadezhin *et al.* (1977, 1978) and Novikov and Polnarev (1980).

The main results of these computations are as follows. The dimensionless amplitude of metric perturbations, $\delta g^{\alpha}_{\beta}$, necessary for the formation of a black hole, is about 0.75–0.9. The uncertainty of the result reflects the dependence on the perturbation profile. Recall that the amplitude of metric perturbation is independent of time as long as $l = ct$ remains much less than the linear size of the perturbed region. If $\delta g^{\alpha}_{\beta}$ is less than 0.75–0.9, the created density perturbations transform into acoustic waves after $l = ct$ increases to about the size of the perturbation.

This answers the first of the questions formulated above.

As for the second question, the computations shows that the black-hole mass at the moment of formation incorporates 10 to 15% of the mass within the scale $l = ct$. This means that the accretion of the gas to the newborn black hole cannot become catastrophic. Computations conform that the gas falling into the black hole from the surrounding space only slightly increases its mass. For an estimate of the possible number of primordial black holes in the Universe, see Novikov *et al.* (1979) and Carr (1983).

13.2. Classical and Quantum Instabilities of White Holes

Formally, the solutions of Einstein's equations that describe black and white holes have a number of similar properties. Thus, the relationship between the solutions describing the formation of a black hole and the explosion of the white hole can be found by using the symmetry of Einstein's equations under time reversal. In fact, the physical properties of black and white holes, including the observational manifestations and the type of interaction with the surrounding matter, are very different. This is not surprising, because the identical behavior of black and white holes would merely signify that the behavior of the surrounding matter and the characteristics of the external observer remain unchanged under the time reversal that transforms white and black holes into each other. However, this is wrong. An

transforms white and black holes into each other. However, this is wrong. An observer invariably moves *forward* in time and receives information on the processes in the field of a hole using *retarded* signals.

A spectacular manifestation of the asymmetry of properties inherent in white and black holes is the instability of the former. The instability of white holes may result both from classical processes due to the interaction with the surrounding matter [Eardly (1974), Frolov (1974), Eardly and Press (1975), Redmount (1984)] and from the processes due to the quantum creation of particles in the gravitational field of the holes [Zel'dovich *et al.* (1974)]. This section will be devoted to describing possible mechanisms of the instability of white holes.

Let us begin with the instability of white holes with respect to ordinary matter falling into them. What is this instability? It is the nonexplosion of the white hole, however strange it may seem (recall the definition of the white hole). The spacetime of an exploding white hole is shown in Figures 8 and 9. Consider an external (i.e. at $r > r_g$) observer long before the white hole explodes. We wish to show that if a small mass of matter, δM, starts to fall into the white hole at a time moment t_0 (for the sake of simplification, we consider the falling of a thin spherical shell), the explosion of the white hole very soon becomes impossible (by the clock of the external observer). Without accretion, the matter of the white hole would expand from the singularity and after some time would emerge from under the gravitational radius (as shown in Figure 8). Now accretion compels the matter to stay inside (the white hole does not explode).

The reason for the instability is as follows.

Let us the trace the motion of the boundary A of the exploding white hole on a Penrose diagram (Figure 86). For the sake of simplification, we assume that the boundary expands at an ultrarelativistic velocity, that is, it is represented by a null geodesic (this assumption does not affect the result). The longer the delay of the explosion, the closer the boundary A lies to the horizon H^+.

Let the mass δM start falling into the hole at a point b (the world line of this mass is shown by the curve B). Let us take into account the back-reaction of the mass δM on the metric. Now the gravitational radius r'_g is

$$r_g' = r_g + 2\ \delta M, \tag{13.2.1}$$

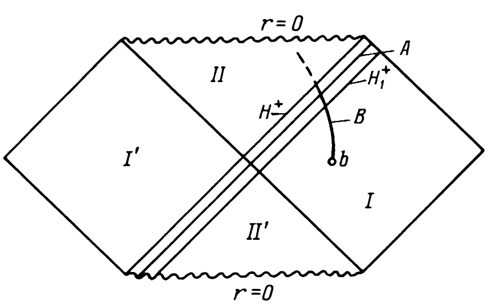


Fig. 86. A scheme clarifying the reason for the instability of the white hole with respect to the accretion of matter from the surrounding space (see text).

where $r_g=2\,M$ is the formal gravitational radius. In view of the change in the metric, the world line of the horizon is $\overset{+}{H_1}$. (The shifting of I^+ and other lines due to the change in the metric are not shown in order to avoid crowding the figure.)

Now it is quite clear if the world line A is to the left of $\overset{+}{H_1}$, the matter of the white hole can never emerge from under the horizon into region I where the observer is (the white hole will never explode).

Let us make several estimates (by order of magnitude). If the mass δM, moving through a nonperturbed spacetime (i.e., spacetime described by nonperturbed r and t), gets closer to r_g than the perturbed horizon r'_g, the explosion of the white hole becomes impossible. Formula (2.3.3) implies that if the mass δM falls from a distance r equal to several r'_g, the fall lasts for

$$\Delta t = t - t_0 \approx \frac{r_g}{c} \ln \frac{r_g}{r - r_g}.$$

Replacing r with r'_g from formula (13.2.1), we obtain an estimate for the interval Δt after which the explosion of a white hole becomes impossible:

$$\Delta t \approx \frac{r_g}{c} \ln \frac{M}{\delta M}. \tag{13.2.2}$$

Obviously, a white hole retains the possibility of exploding for a short time, even if δM is extremely small.

Accretion of matter into white holes causes their instability and ultimately transforms them into a type of black hole. The fate of such a white holes must, therefore, be discussed together with that of black holes. We will return to this situation in the next section.

Frolov (1974) analyzed the changes in the motion of the expanding matter of a white hole when it collides with matter in the T_+-region (region II' in Figure 86).

We will now consider quantum instability of white holes [Zel'dovich *et al.* (1974)]. This instability arises because the particles that are created with high intensity near the Schwarzschild singularity of the white hole, move outward in the T_+ -region and, therefore, can strongly influence the metric far from the singularity. They can also emerge from under the gravitational radius and, thus, reduce the mass the white hole.

It is found that any change due to particles created in the white hole stands in the way of the explosion of the retarded core.

Finally, another aspect of the problem arises because a white hole must have existed not in empty space but in the earliest expanding Universe. This means that the surrounding matter interacted actively with the white hole and the particles created therein at the earliest stages of the cosmological expansion.

Let us begin by analyzing the creation of particles in the neighborhood of Schwarzschild singularity in the T_+-region. Consider an 'eternal' white hole (see Section 2.7). The singularity in it is space-like and homogeneous. Hence, the centroids of each volume element of the created particles must be at rest in a reference frame with homogeneous space. The general form of such a reference frame

(taking into account the effect of the created particles on the metric) in the spherically symmetric case is [Novikov (1964b)]

$$ds^2 = -dt^2 + e^{\lambda} dR^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2), \tag{13.2.3}$$

where λ and r are functions only of t.

We choose t so that the singularity is at $t = 0$. As $t \to 0$,

$$r \propto t^{2/3}, \ e^{\lambda} \propto t^{-2/3}. \tag{13.2.4}$$

In such a metric, particles are created close to the singularity, perhaps at $t \approx t_{Pl}$ [Zel'dovich and Starobinsky (1971)]. The energy density of these particles is

$$\varepsilon_{Pl} \approx \frac{1}{G t_{Pl}^2}. \tag{13.2.5}$$

At $t > t_{Pl}$, the rate of particle creation drops sharply and can be neglected. Later on, the density decreases because of volume expansion. If the equation of the state of the newborn matter is known, then the evolution of the system can be calculated. Zel'dovich *et al.* (1974) constructed models for a number of equations of state. Not all of these models are realistic but they possess a number of common properties that reflect the specifics of the respective problems and sometimes make it possible to completely solve the problem.

The simplest (nonrealistic) assumption is that the pressure of the created particles is zero ($p = 0$). The solution is written in parametric form

$$r = \frac{1}{2} r_g (1 - \cos\xi),$$

$$e^{\lambda/2} = \text{cotan}\,\frac{\xi}{2} + \alpha\left(1 - \frac{1}{2}\,\text{cotan}\,\frac{\xi}{2}\right),$$

$$t = \frac{1}{2} r_g (\xi - \sin\xi), \tag{13.2.6}$$

$$8\pi G\varepsilon = \alpha r^{-2} e^{-\lambda/2},$$

where

$$\alpha \approx \frac{r_g}{r_{Pl}} \gg 1, \ \ 0 \le \xi \le 2\pi, \ -\infty < R < \infty.$$

The solution describes the uniform expansion of the mass of newborn particles from the moment $t \approx t_{pl}$ to the moment t_1 at which $r = r_g$ and energy density is $\varepsilon = (8\pi G r_g^2)^{-1}$, followed by a subsequent compression of the matter to the singularity.

In order to clarify the physical meaning of the solution, we demand that the following condition be satisfied: Particles are created at nearly $t \approx t_{pl}$, on the interval

of R from $-\infty$ to some R_1, with no particles created at $R > R_1$. (Later we will show how to make this assumption realistic.) The structure of the spacetime then has the form shown in Figure 87. The entire mass of the created particles lies under the gravitational radius and does not leave the white hole.

We now assume that we have to analyze, not an 'eternal' white hole but a hole with a retarded expansion of the core. It is not difficult to show that the particles created close to the Schwarzschild singularity do not allow this core to escape from under the gravitational radius.

Indeed, a prolonged retardation of the expansion indicates that the boundary of the core must lie at $R = R_2$ at the moment $t \approx t_{\rm Pl}$ (close to $r = 0$), far to the left of the point R_1 ($R_2 \gg R_1$). A signal leaving R_2 at a moment $t_{\rm Pl}$ and travelling rightward to R_1, covers only a finite distance ΔR during the entire time of expansion t_1. Estimates show that

$$\Delta R \approx 7 r_{\rm Pl} \ll r_g. \tag{13.2.7}$$

If $R_1 - R_2 \gg \Delta R$, the signal does not reach R_1 by the time t_1 when expansion is terminated. Not only will the matter of the retarded core be unable to escape to a distant observer, but no signals emitted by the explosion will escape from the white hole and reach R_1. The retarded core will be buried under the mass of particles created.

Now we can discuss the assumption of no particle creation near $r = 0$ to the right of the coordinate R_1. One has to remember that the white hole evolves not in an empty space but in the expanding Universe [Novikov (1964b), Ne'eman (1965)]. If the matter of homogeneous cosmological model surrounding the white hole is located to the right of R_1 close to $r = 0$, there is almost no particle production in this region [under standard assumptions; see Zel'dovich and Starobinsky (1971)]. If we also assume that the pressure in the surrounding matter is zero (which is not realistic), the matter produces no effect on the region to the left of R_1. The structure of spacetime in this model is shown in Figure 88.

Now we will look at more realistic models.

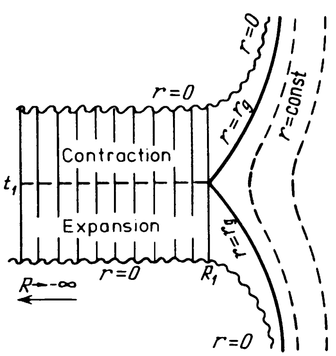


Fig. 87. Schematic presentation of the expansion and compression of matter at $p = 0$ inside a black hole. The matter is created via quantum processes close to $r = 0$ to the left of R_1.

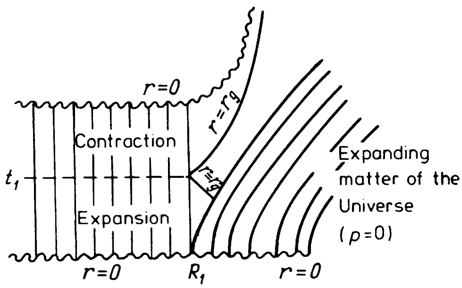


Fig. 88. A white hole with the created matter ($p = 0$) in the cold Universe model (with matter also at $p = 0$).

Assume that the newborn particles do not interact with one another and thus form two colliding beams that travel along the radial coordinate with the speed of light. In this case, $-T_0^0 = T_1^1 = \varepsilon$, the remaining $T_i^k = 0$. Another assumption is that the interaction of the created particles produces Pascal pressure $p = \varepsilon/3$. The solutions of these cases are similar to (13.2.6) for $p = 0$; see Zel'dovich *et al.* (1974). They also describe the expansion of the system to some $r_{\max}$ and the subsequent contraction to a singularity. Here again the signal travelling at a speed of light covers only a finite small distance along R over the system's entire expansion time. This is why the created particles do not allow a retarded-expansion core (as in the case of $p = 0$) to explode and expand until it reaches the external observer. An essential difference in comparison with the case of $p = 0$ is that a matter flux to the right across the boundary R_1 arises if $T_1^1 \neq 0$. This flux may escape from under r_g and thereby reduce the mass of the white hole.

If the white is in a cold Universe with matter at zero pressure ($p = 0$), then the reduction in the white-hole mass, due to the spontaneous leaking of the newborn matter from the hole can be quite high, as was shown in the already-cited paper by Zel'dovich *et al.* (1974).

However, if the white hole is considered in a real hot Universe with matter, and its equation of state is $p=\varepsilon/3$, the situation is changed. The pressure of the surrounding hot matter constrains the leaking of the new matter from the white hole. In this case it is very likely that the loss of mass due to leaking is considerably lowered. We will not go into the details of this phenomenon, since it is more a problem of cosmology (on the accretion of matter onto compact cores in the hot Universe, see Section 13.1).

13.3. What is Left after a Quantum Decay of a Black Hole?

Unfortunately, it is not possible to date to give an unambiguous answer to this question. The point is that any attempt to solve this problem brings one face to face with other problems that belong in the realm of quantum gravity. Theoretically, quantum gravity appears to be quite far from completion, while the difficulties

inherent in it (divergences, nonrenormalizability, ambiguity of going off the mass shell, the possibility of changes of spacetime topology) are fundamental. As a result, there is no complete self-consistent quantum theory of evaporating black holes. A natural approach to this situation is to analyze models that reflect specific facets of the complete problem.

We limit the analysis to the spherically symmetric case.[2] The corresponding average metric $g_{\mu\nu} = \langle \hat{g}_{\mu\nu} \rangle$ is conveniently written in the form [Bardeen (1981)]

$$ds^2 = -e^{2\psi} F\, dv^2 + 2\, e^{\psi}\, dr\, dv + r^2\, d\omega^2. \tag{13.3.1}$$

Here v is the null coordinate of the advanced time and ψ and F are functions of r and v with the following invariant meaning:

$$F(r, v) \equiv g^{\mu\nu} r_{,\mu} r_{,\nu}, \quad e^{-\psi(r,v)} = g^{\mu\nu} r_{,\mu} v_{,\nu}. \tag{13.3.2}$$

Assume that the spacetime is asymptotically flat and demand that

$$\lim_{r\to\infty} F(r, v) = 1, \quad \lim_{r\to\infty} \psi(r, v) = 0. \tag{13.3.3}$$

Of course, the range of applicability of geometry descriptions in terms of the averaged metric $g_{\mu\nu} = \langle g_{\mu\nu} \rangle$ is limited. Thus, it is not valid on scales less than $l_{\rm Pl}$. owing to the strong quantum fluctuations of the gravitational field. Later we will return to a discussion of a possible role of fluctuation ($r < l_{\rm Pl}$) but for now we will outline some general properties of the averaged metric.

Essential information on the properties of spacetime can be obtained by analyzing the behavior of the surface $F = \text{const}$ of the function F. Thus, the exterior part of the surface $F = 0$ coincides with the apparent horizon. If the created black hole were static, the apparent horizon would coincide with the event horizon and the surface $F = 0$ would be described by the equation $r = 2M$, where M is the mass of the created black hole. The quantum evaporation of the black hole makes the horizon nonstatic, with the size decreasing with time (curve BC in Figure 89). If $r = \rho(v)$ is the equation of the outgoing radial null rays, we find that on the $F = 0$ surface,

$$\frac{d\rho}{dv} = e^{\psi} F = 0, \quad \frac{d^2\rho}{dv^2} = (e^{\psi} F)_{,v}\,. \tag{13.3.4}$$

Thus, $d^2\rho/dv^2 > 0$ on the segment BC.

Using expression (13.3.1) for the metric, we can calculate the corresponding Ricci tensor and verify that in this general case the metric satisfies Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} \tag{13.3.5}$$

with a nonzero right-hand side. In particular,

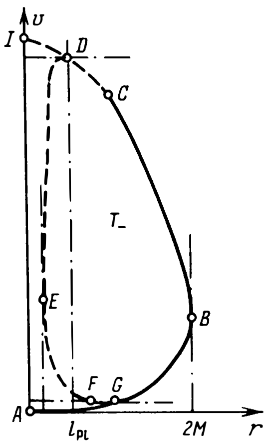


Fig. 89. Versions of the possible behavior of the apparent horizon in the quantum evaporation of a black hole.

$$T_{vv} = \frac{1}{8\pi r}\left\{ e^{\psi}F\left[\frac{e^{\psi}}{r}\partial_r(r(1-F)) + \psi_{,v}\right] - (e^{\psi}F)_{,v}\right\}. \tag{13.3.6}$$

This relation simplifies for the surface $F = 0$ (the apparent horizon):

$$T_{vv}\big|_{F=0} = -\frac{1}{8\pi r}(e^{\psi}F)_{,v}\big|_{F=0} = -\frac{1}{8\pi r}\frac{d^2\rho}{dv^2}\big|_{F=0}. \tag{13.3.7}$$

It shows that there is a flux of negative energy density across the segment BC of the apparent horizon, in complete agreement with the results presented in Chapter 10.

For describing the processes over the entire time interval v during which the black hole mass $m(v)$ [e.g., we can choose $m(v) = r/2|_{F=0}$] is much greater than the Planck mass m_{Pl} and, hence, the rate of change of the apparent horizon size, $d(r|_{F=0})/dv$, is small in comparison with the speed of light, one can use the *quasistatic approximation* [Hajicek and Israel (1980), Bardeen (1981), Frolov (1981), Nityanada and Narayan (1981)].[3]

The last stage of evaporation at which the mass of the black hole becomes comparable to the Planck mass is the most difficult one to describe. The spacetime curvature near the apparent horizon may reach $1/l_{\mathrm{Pl}}^2$ at this stage so that in the general case the mean metric can be found only if the effective action, covering all quantum corrections, is known. In the general case, it is legitimate to state that if the surface $F = 0$ intersects $r = 0$, a singularity arises in which curvature invariants tend to infinity [Frolov and Vilkovisky (1979, 1981), Kodama (1979, 1980)].

In principle, it is possible to avoid the creation of a naked singularity by assuming that the $F = 0$ surface is closed, never intersecting the line $r = 0$ (line

BCDEFG in Figure 89). In this case, the singularity inside the black hole also disappears.[4] This possibility was discussed by Frolov and Vilkovisky (1979, 1981, 1982), Thomboulis (1980), and Hasslacher and Mottola (1981). The spacetime near $r = 0$ is locally flat for this solution and we may expect that its curvature at $r \lesssim l_{Pl}$ is of the order l_{Pl}^{-2}, with the internal part of the line $F = 0$ (*FED*) being spaced from $r = 0$ by a distance of the order l_{PL}. It can be shown, using general relation (13.3.7), that $T_{\upsilon\upsilon} < 0$ on the segment *EDB* and $T_{\upsilon\upsilon} \gtrsim 0$ on the segment *EFGB*, where *E* and *B* are the points at which $F = 0$ is tangent to the lines $r =$ const [Roman and Bergman (1983)].

A spacetime with closed horizon $F = 0$ does not have an event horizon. Rigorously speaking, a black hole does not exist in this situation. However, a region that does not let any signals out exists throughout the time of quantum evaporation. If the initial mass of such an object is much greater than the Planck mass, its manifestations are indistinguishable from those of a black hole for a very long time.

A number of fundamental questions arise when this model of a singularity-free 'black hole' is analyzed. One of them concerns the conservation of the baryon charge in this system. Assume that a collapsing system has a considerable baryon charge. This charge cannot change substantially in the process of quantum evaporation, owing to the symmetry of baryon and antibaryon creation.[5] Then again, if this 'black hole' completely evaporates, the original baryon charge disappears. We thus come to an obvious violation of the baryon charge conservation.

This situation could be regarded as a difficulty inherent to the model if processes that do not conserve baryon charge were impossible. In fact, such nonconserving processes are widely discussed now in connection with Grand Unification Theories; they involve supermassive (with masses of $10^{14} - 10^{15}$ GeV) vector *X* and *Y* bosons. When the collapsing matter compresses to a density $\rho \sim 10^{74} - 10^{78}$ g/cm^3, corresponding to the mass of these particles, the system almost immediately becomes neutral with respect to the baryon charge, regardless of its initial value.[6] As a result, the matter can get rid off its original baryon charge, even before the Planck density $\rho_{Pl} \sim 10^{94}$ g/cm^3 is reached.[7]

The motion of particles and light beams in a spacetime with a closed horizon $F = 0$ is characterized by a number of unusual features. Particles falling along radii, cross the T_- -region in a short proper time on the order of r_g/c, reach the line $r = 0$, and start moving away from the center. However, they cannot cross the line *ED* again and enter the T_- -region, Therefore, all such particles accumulate close to *ED* (in the classical description) and escape to the external space after the 'black hole' has evaporated (in a proper time of the order r_g/c). The particles have a 'blue shift' $\propto \exp(\kappa_- V_{BH})$, where

$$\kappa_- = \frac{1}{2} \left| \frac{\partial(e^{\psi} F)}{\partial r} \right|_{F=0} \tag{13.3.8}$$

is an analogue of the surface gravity for the interior horizon (on the line *ED*) and V_{BH} is the lifetime of the 'black hole'. A similar 'blue shift' effect must take place for waves trapped in such a 'black hole'. In quantum analysis, this effect results in the extremely intensive creation of particles in the decay of the 'black hole'. As this energy release cannot be higher than a quantity on the order of the Planck mass (otherwise the energy conservation law in the external space would be violated), one can conclude that the surface gravity κ_- must be less or of the order of V_{BH}^{-1} provided the estimate of radiation based on using quantum theory in a given averaged metric is correct [Bolashenko and Frolov (1984, 1986)].[8]

The absence of an event horizon in a model with a closed horizon could lead to another, extremely interesting corollary.

The creation of an outgoing particle in a black hole is accompanied by a particle emerging inside it. A distant observer records only a part of the particles; hence, the radiation of the black hole possesses entropy and is described by a density matrix (Section 9.3). There is no event horizon in the model with a closed horizon, so that the particles created inside the 'black hole' can leave it after evaporation is completed. As a result, the quantum state could again be pure from the standpoint of the distant observer. In other words, the growth of entropy in the external space due to the thermal radiation of the black hole at a stage when its mass is still much greater than the Planck mass, is predicted as being replaced by a sharp drop to zero at the last stage of decay.

The analysis above operated in terms of the approximation in which the created particles are assumed noninteracting and the fluctuations of the gravitational field are neglected. Both these assumptions seems to fail for the description of the propagation of particles in the region close to the internal horizon *ED*. The interactions between particles inside a black hole and the scattering of particles on gravitational field fluctuations are such that particles may 'forget' their phases [9] and no entropy decrease occurs when the black hole decays.

In addition to the scenarios discussed above (the formation of a naked singularity and a model with a closed horizon in which a black hole burns out completely), another version is possible in which a residue is left after the black hole has evaporated. An *elementary black hole* of a mass on the order of one Planck mass is a possible form of this residue.[10] (In Figure 89, this case would correspond to the following behavior of the $F = 0$ line: The inner and outer parts of this line continue infinitely along the coordinate v, getting very close to each other or even merging.) An analysis of the spherically symmetric collapse of a system of less than the Planck mass shows that quantum effects, including the vacuum polarization, make the 'averaged metric' $g_{\mu\nu} = \langle \hat{g}_{\mu\nu} \rangle$ that describes the geometry as everywhere regular and precludes the formation of the apparent horizon (and, hence, of the event horizon) [Frolov and Vilkovisky (1979, 1981, 1982)]. This result indicates that black holes of less than the Planck mass cannot exist; that is, if elementary black holes exist at all, their masses must be in order of the Planck mass. The properties of such objects, called the maximons, were first discussed by Markov (1965, 1966) [see also Hawking (1971a)].

To complete the discussion of the internal structure of black holes and of their ultimate fate, we will mention one additional possibility. It has already been mentioned that equations describing the gravitational field are expected to be considerably modified when curvature becomes comparable to the Planckian value. If this modification is such that the curvature can never exceed the limiting Planckian curvature, and the effective equation of state at the limiting curvature has vacuum-like form ($R_u^v = \Lambda \delta_u^v$), a closed de Sitter world can form inside the black hole [Frolov, Markov, and Mukhanov (1988)]. After the stage of contraction (the deflation stage), this world may begin to expand (the inflation stage) and give birth to a new macroscopic universe.

13.4. Elementary Black Holes (Maximons). Virtual Black Holes and the Foam-Like Structure of Spacetime

The problem of the stability of maximons with respect to the quantum decay is one of the basic problems for the hypothesis of maximon existence. Formally, the black hole temperature vanishes if its parameters – the electric (Q) and magnetic (P) charges [11] and the angular momentum (J) – are related to the mass M of the black hole by the formula

$$M^2 = \frac{Q^2 + P^2}{G} + \frac{J^2 c^2}{G^2 M^2}. \tag{13.4.1}$$

Owing to the quantum effect of vacuum polarization, the modification of the Einstein–Maxwell equations may change conditions (13.4.1) for the vanishing of the black-hole temperature. The possibility that the temperature will vanish for the neutral maximon as well, if these effects are taken into account, cannot be excluded. Unfortunately, the problem of stability of such a black hole with respect to quantum processes remains unsolved. The point is that maximons (assuming that they exist) have the minimal mass admissible for black holes; a loss of arbitrarily small mass results in their complete decay. It is logical that this process should yield quanta with a characteristic energy $\varepsilon \sim m_{\text{Pl}} c^2$, whose wavelength $\lambda \sim \hbar c/\varepsilon$ is comparable to their gravitational radius. Presumably, the approximation based on assuming that newborn particles exert a negligible effect on the metric is unacceptable under these conditions.

On the whole, the answer to the question of the existence and stability of maximons involves the behavior of physical interaction at energies comparable to the Planck value. The final solution can be expected only after the theory of quantum gravity has been constructed. Probably, a natural solution will be found in the framework of the unified theory of all interactions (based, e.g., on one of the versions of supergravity or string theory).

If nature has elementary black holes, they possess a number of fascinating properties [Markov (1966)]. They are characterized by extremely small interaction cross-section on the order of $10^{-66}\,\text{cm}^2$. When a maximon falls in the gravitational

field of the Earth, it gains energy in the order of 10^{20} eV. However, it seems to be impossible to observe maximons using their ionizing ability, even if they are charged and the interaction with matter is sufficiently strong, because of their low velocities. Maximons are difficult to retain and accumulate in ordinary matter on the surface of the Earth, because the terrestrial gravitational field imparts energy $\sim 10^3$ eV on the length of the intermolecular spacing, which is much more than the energy of the intermolecular interaction.

As a result of the small cross-section of interaction between matter and neutral maximons, a considerable (and even predominant) part of matter in the Universe could consist of maximons. Thus, maximons could act as the the dark matter which is now recognized as a reality in cosmology [Markov (1981b)].[12]

It appears that the most promising method of searching for maximons is based on recording their decay products. If one assumes the existence of a bound state of many maximons [Markov and Frolov (1979)] or of a small number of them, for instance, of a maximon pair [Aman (1983)], a merger of a pair of maximons into one can be expected in the evolution of such systems, with energy release on the order of 10^{28} eV. Processes of this type could presumably by recorded in DUMAND-type experiments [for details, see Markov (1981a) and Markov and Zheleznykh (19810].

Stable maximons may play the role of the heaviest possible fundamental particles [Markov (1976, 1981c)]. If we relate the size of a particle with a Compton wavelength $\lambda = \hbar/mc$, this size is found to be smaller than the gravitational radius for particles of masses $m > m_{\rm Pl}$.[13] Elementary black holes, even if they are unstable, are important for particle theory for at least one more reason. Indeed, when calculations are carried out in modern quantum theory (for example, the calculations of the self-energy of particles), it is usually necessary to take into account the contribution of the intermediate state with an arbitrarily high energy; the result is the familiar divergences. If the gravitational interaction of the appropriate virtual particles is included in the calculations, and the possibility of the appearance of *virtual* (short-lived) black holes in the intermediate state is taken into account, these divergences may be removed [Markov (1971)].

Virtual black holes may also appear in the vacuum as a result of quantum fluctuations. Quantum fluctuations of the gravitational field are the greater, the shorter the length scale. The fluctuations of the metric are comparable to the metric itself over distances on the order of the Planck length. Such fluctuations signal the possibility of strong deviations from the flat geometry and Euclidean topology. In other words, continuous creation and annihilation of virtual black holes makes the spacetime on small scales resemble a soapy foam.

The concept of a foam-like (cell) structure of spacetime, formulated by Wheeler in the 1950s, was recently developed by Hawking and his coworkers [Hawking (1978, 1984), Hawking *et al.* (1979, 1980), Warner (1982)].

Some of the interesting applications of these ideas deserve special mention: (1) possible violation of quantum coherence and the transformation of the pure state into a mixed one as a result of the interaction of the quantum field with the fluctuations of the gravitational field [Hawking (1984, 1988) and Lavrelashvili *et al.*

(1988)], and (2) nonconservation of the baryon and lepton numbers in the process of interaction between elementary particles and virtual black holes (the space 'foam') [Hawking (1984)]. Even though the proton lifetime with respect to this process is many orders of magnitude greater than that predicted in Grand Unification Theories, the very possibility of such processes may by of fundamental importance, especially in the discussion of the origin of the Universe.

Conclusion

Some 20 years ago, very few scientists thought that black holes may really exist. Attention focussed on the black-hole hypothesis after neutron stars had been discovered. It was rather surprising that astrophysicists immediately 'welcomed' them. They found their place not only in the remnants of supernova explosions but also in the cores of globular clusters, galaxies, and quasars.

The question about the cosmological role of small black holes became especially important after Hawking's discovery of the quantum evaporation of black holes. The hypothesis of elementary black holes (maximons) is intersecting not only in its cosmological consequences but also in the aspects concerning particle physics. In all likelihood, virtual black holes will be an important element of the future quantum theory of gravitation. The investigation of the properties of black holes revealed profound relationships between gravitation, quantum theory, and thermodynamics. This progress (and especially the fact that the theory of black holes participating in physical processes requires qualitatively new ideas) has brought about the growth of an essentially novel branch of physics during the part 10 to 15 years: black-hole physics with its own object of study and its inherent problems. These problems are often of a very fundamental nature while the object is so astonishing that this new field attracted quite a few researchers. We wanted this book to explain the main phenomena in black-hole physics. We fully understand that some aspects of this field deserve a more detailed presentation than we were able to provide. We feel slightly justified because incompleteness frequently reflects the current situation in the theory. Black-hole physics is a young and rapidly progressing science. We hope that this development will not only clarify the puzzling aspects but also make physicists happy with new, even less predictable results.

Appendix

In this Appendix, we have collected the most important formulas of Riemannian geometry and general relativity that are used throughout the book. We only list the main relations and give very brief comments, because the derivations of these formulas and all necessary clarifications can be found in the available textbooks and monographs [see, e.g., Landau and Lifshitz (1975), Misner, Thorne, and Wheeler (1973), Hawking and Ellis (1973), Kramer *et al.* (1980), Vladimirov (1982)].

Indices: Greek indices α, β,... run through the values 0,1,2,3; Roman indices i, j,... run through the values 1, 2, 3.

Symmetrization $A_{(\mu_1 \ldots \mu_p)}$ and *antisymmetrization* $A_{[\mu_1 \ldots \mu_p]}$ of a tensor $A_{\mu_1 \ldots \mu_p}$:

$$A_{(\mu_1 \ldots \mu_p)} = \frac{1}{p!} \sum_{\substack{\text{over all} \\ \text{permutations}}} A_{\mu_1 \ldots \mu_p},$$

$$A_{[\mu_1 \ldots \mu_p]} = \frac{1}{p!} \sum_{\substack{\text{over all} \\ \text{permutations}}} (-1)^J A_{\mu_1 \ldots \mu_p},$$

where $J = 0$ if the permutation is even, and $J = 1$ otherwise.
The *metric* of spacetime, $ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta$, has a signature $- + + +$.

A *smooth curve* $x^\mu(\lambda)$ is said to be space-, time-, or light-like at a point $\lambda = \lambda_0$ if the vector $u^\mu = dx^\mu/d\lambda$ tangent to it at this point satisfies the condition $u^\mu u_\mu > 0$, $u^\mu u_\mu < 0$, or $u^\mu u_\mu = 0$, respectively. The curve is said to be causal if $u^\mu u_\mu \leq 0$ at each of its points.

Causal future $J^+(Q)$ [*causal past* $J^-(Q)$] of a set Q is the set of points for each of which there is a past-directed (future-directed) causal curve that intersects Q.

Future Cauchy domain $D^+(Q)$ [*past Cauchy domain* $D^-(Q)$] of a set Q is the set of points such that any past-directed (future-directed) causal curve passing through it intersects Q.

Total Cauchy surface is a nontime-like hypersurface that is intersected by each causal curve exactly once.

Riemannian curvature tensor:

$$R^\mu{}_{\nu\alpha\beta} = \partial_\alpha \Gamma^\mu_{\nu\beta} - \partial_\beta \Gamma^\mu_{\nu\alpha} + \Gamma^\mu_{\sigma\alpha} \Gamma^\sigma_{\nu\beta} - \Gamma^\mu_{\sigma\beta} \Gamma^\sigma_{\nu\alpha} \tag{A.1}$$

where

$$\Gamma^\mu{}_{\alpha\beta} = g^{\mu\nu} \Gamma_{\nu,\alpha\beta}, \quad \Gamma_{\nu,\alpha\beta} = \frac{1}{2} (g_{\alpha\nu,\beta} - g_{\alpha\beta,\nu} + g_{\beta\nu,\alpha}) \tag{A.2}$$

Ricci tensor:

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} \, . \tag{A.3}$$

Weyl tensor:

$$C_{\mu\nu\sigma\tau} = R_{\mu\nu\sigma\tau} + g_{\nu[\sigma}S_{\tau]\mu} - g_{\mu[\sigma}S_{\tau]\nu}, \tag{A.4}$$

where

$$S_{\mu\nu} = R_{\mu\nu} - \frac{1}{6} g_{\mu\nu} R .$$

Einstein tensor:

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R .$$

Covariant derivative:

$$\nabla_{\alpha} B^{\beta_1 \dots}{}_{\gamma_1 \dots}$$

$$= \partial_{\alpha} B^{\beta_1 \dots}{}_{\gamma_1 \dots} + \Gamma^{\beta_1}{}_{\alpha\mu} B^{\mu \dots}{}_{\gamma_1 \dots} + \dots - \Gamma^{\mu}{}_{\alpha\gamma_1} B^{\beta_1 \dots}{}_{\mu \dots} \dots \, . \tag{A.5}$$

Another notation for the ordinary four-dimensional covariant derivative: $\nabla_{\alpha}(\ ) \equiv (\ )_{;\alpha}$. Covariant derivatives with respect to the three-dimensional metric are denoted by $\nabla_i(\ ) = (\ )_{:i}$. The notation used for two-dimensional covariant derivatives is $(\ )_{|A} (A = 1,2)$.

Commutator of covariant derivatives:

$$(\nabla_{\alpha}\nabla_{\beta} - \nabla_{\beta}\nabla_{\alpha}) B_{\mu \dots}{}^{\nu \dots} = R_{\alpha\beta\mu}{}^{\sigma} B_{\sigma \dots}{}^{\nu \dots} + \dots - R_{\alpha\beta\sigma}{}^{\nu} B_{\mu \dots}{}^{\sigma \dots} - \dots \, . \tag{A.6}$$

Lie derivative $\mathfrak{L}_{\xi} A^{\alpha \dots}{}_{\beta \dots}$ of a tensor field $A^{\alpha \dots}{}_{\beta \dots}$ along a vector field ξ is given by the relation

$$\begin{aligned} \mathfrak{L}_{\xi} A^{\alpha \dots}{}_{\beta \dots} &= \xi^{\mu} \partial_{\mu} A^{\alpha \dots}{}_{\beta \dots} - \partial_{\mu} \xi^{\alpha} A^{\mu \dots}{}_{\beta \dots} - \dots + \partial_{\beta} \xi^{\mu} A^{\alpha \dots}{}_{\beta \dots} + \dots \\ &= \xi^{\mu} \nabla_{\mu} A^{\alpha \dots}{}_{\beta \dots} - \nabla_{\mu} \xi^{\alpha} A^{\mu \dots}{}_{\beta \dots} - \dots + \nabla_{\beta} \xi^{\mu} A^{\alpha \dots}{}_{\beta \dots} + \dots , \end{aligned} \tag{A.7}$$

$$\mathfrak{L}_{\xi} \eta^{\alpha} \equiv [\xi, \eta]^{\alpha} = \xi^{\mu} \partial_{\mu} \eta^{\alpha} - \eta^{\mu} \partial_{\mu} \xi^{\alpha}, \tag{A.8}$$

$$\mathfrak{L}_{\xi} \mathfrak{L}_{\eta} - \mathfrak{L}_{\eta} \mathfrak{L}_{\xi} = \mathfrak{L}_{[\xi, \eta]} . \tag{A.9}$$

Fermi–Walker derivative $\mathfrak{F}_{\xi} A^{\alpha \dots}{}_{\beta \dots}$ of the tensor field $A^{\alpha \dots}{}_{\beta \dots}$ along a vector field ξ $(\xi^{\mu} \xi_{\mu} \neq 0)$:

$$\mathfrak{F}_\xi A^{\alpha\ldots}{}_{\beta\ldots} = \xi^\mu \nabla_\mu A^{\alpha\ldots}{}_{\beta\ldots} + \mathfrak{F}^\alpha{}_\sigma A^{\sigma\ldots}{}_{\beta\ldots} + \ldots - \mathfrak{F}^\sigma{}_\beta A^{\alpha\ldots}{}_{\sigma\ldots} - \ldots, \tag{A.9a}$$

where

$$\mathfrak{F}^\alpha{}_\sigma = (w^\alpha \xi_\sigma - \xi^\alpha w_\sigma)\varepsilon(\xi), \quad w^\alpha = \xi^\beta \xi^\alpha{}_{;\beta}, \quad \varepsilon(\xi) = \mathrm{sign}(-\xi^\mu \xi_\mu).$$

Parallel transport. A tensor field $A^{\alpha\ldots}{}_{\beta\ldots}$ undergoes parallel transport along a vector field ξ if the condition

$$\xi^\mu \nabla_\mu A^{\alpha\ldots}{}_{\beta\ldots} = 0 \tag{A.10}$$

is satisfied. The parallel transport along ξ is said to be *in the Lie sense* if

$$\mathfrak{L}_\xi A^{\alpha\ldots}{}_{\beta\ldots} = 0, \tag{A.11}$$

and *in the Fermi–Walker sense* if

$$\mathfrak{F}_\xi A^{\alpha\ldots}{}_{\beta\ldots} = 0. \tag{A.12}$$

A geodesic $x^\alpha(\lambda)$ is defined as the solution of the equation

$$\frac{d^2 x^\alpha}{d\lambda^2} + \Gamma^\alpha{}_{\mu\nu} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = f(\lambda)\frac{dx^\alpha}{d\lambda}, \tag{A.13}$$

where $f(\lambda)$ is an arbitrary function. This function can be set to zero by changing the parametrization $\tilde{\lambda} = \tilde{\lambda}(\lambda)$. The corresponding parameter is said to be affine. The affine parameter is defined up to a linear transformation. The affine parameter for timelike

$$\left(g_{\mu\nu} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} < 0 \right)$$

and spacelike

$$\left(g_{\mu\nu} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} > 0 \right)$$

geodesics is proportional to the proper length $\int |ds^2|^{1/2}$ along the curve.

Geodesic deviation equation. Let $n^\mu(\lambda)$ be a vector that connects two nearby geodesics for the identical value of the affine parameters λ along them. Then the equation

$$\frac{D^2 n^\alpha}{d\lambda^2} + R^\alpha{}_{\beta\gamma\delta} u^\beta n^\gamma u^\delta = 0,$$

where

$$u^\beta = \frac{dx^\beta}{d\lambda}, \qquad \frac{D}{d\lambda} = u^\mu \nabla_\mu,$$

is valid.

Killing vector field ξ^μ in the space with metric $g_{\mu\nu}$ is defined by the relation

$$\mathcal{L}_\xi g_{\mu\nu} \equiv 2\xi_{(\mu;\nu)} = 0. \tag{A.14}$$

The Killing vector field ξ^μ satisfies the equation

$$\xi_{\mu;\alpha;\beta} = R_{\gamma\beta\alpha\mu}\xi^\gamma. \tag{A.15}$$

If ξ^μ and η^μ are two Killing vector fields, then $[\xi, \eta]^\mu \equiv \xi^\alpha \partial_\alpha \eta^\mu - \eta^\alpha \partial_\alpha \xi^\mu$ is also a Killing vector field.

If ξ^μ is time-like ($\xi^\mu \xi_\mu < 0$) and $u^\mu = \xi^\mu / |\xi_\alpha \xi^\alpha|^{1/2}$ is the four-velocity of motion along ξ^μ, then the acceleration is

$$w^\mu = u^\alpha \nabla_\alpha u^\mu = \frac{1}{2}\nabla^\mu \ln |\xi_\alpha \xi^\alpha|. \tag{A.16}$$

A Killing tensor field is a symmetric tensor field $\xi_{\alpha_1 \ldots \alpha_m} = \xi_{(\alpha_1 \ldots \alpha_m)}$ satisfying the condition

$$\xi_{(\alpha_1 \ldots \alpha_m;\beta)} = 0. \tag{A.17}$$

Conformal transformations are defined as metric transformations of the type

$$g_{\alpha\beta}(x) = \Omega^2(x)\hat{g}_{\alpha\beta}(x). \tag{A.18}$$

A tensor field $A^{\alpha_1 \ldots \alpha_n}{}_{\beta_1 \ldots \beta_m}$ is a field of conformal weight s if it transforms under (A.18) according to the formula

$$A^{\alpha_1 \ldots \alpha_n}{}_{\beta_1 \ldots \beta_m} = \Omega^{s-n+m}\hat{A}^{\alpha_1 \ldots \alpha_n}{}_{\beta_1 \ldots \beta_m}. \tag{A.19}$$

If $\hat{\nabla}_\gamma$ is a covariant derivative in a metric $\hat{g}_{\alpha\beta}$, then

$$\hat{\nabla}_\gamma A^{\alpha\ldots}{}_{\beta\ldots} = \nabla_\gamma A^{\alpha\ldots}{}_{\beta\ldots} + C^\alpha_{\gamma\sigma} A^{\sigma\ldots}{}_{\beta\ldots} + \ldots - C^\sigma_{\gamma\beta} A^{\alpha\ldots}{}_{\sigma\ldots} - \ldots, \tag{A.20}$$

where

$$C^\alpha_{\gamma\beta} = C^\alpha_{(\gamma\beta)} = -\Omega^{-1}[\delta^\alpha_\beta \nabla_\gamma \Omega + \delta^\alpha_\gamma \nabla_\beta \Omega - g_{\gamma\beta} g^{\alpha\sigma} \nabla_\sigma \Omega]. \tag{A.21}$$

A Weyl tensor $C_{\alpha\beta\gamma}{}^\delta$ is invariant under conformal transformations, while the curvatures $R_{\alpha\beta\gamma}{}^\delta, R_{\alpha\beta}$ and R are transformed as follows:

$$\hat{R}_{\alpha\beta\gamma}{}^\sigma = R_{\alpha\beta\gamma}{}^\sigma - 2\nabla_{[\alpha} C^\sigma_{\beta]\gamma} + 2C^\lambda_{\gamma[\alpha} C^\sigma_{\beta]\lambda}, \tag{A.22}$$

$$\hat{R}_{\alpha\beta} = R_{\alpha\beta} + 2\Omega^{-1}\nabla_\alpha \nabla_\beta \Omega + \Omega^{-1} g_{\alpha\beta} \nabla^\sigma \nabla_\sigma \Omega -$$

$$-3\Omega^{-2}g_{\alpha\beta}g^{\gamma\sigma}\nabla_\gamma\Omega\nabla_\sigma\Omega, \tag{A.23}$$

$$\hat{R} = \Omega^2 R + 6\Omega g^{\gamma\sigma}\nabla_\gamma\nabla_\sigma\Omega - 12g^{\gamma\sigma}\nabla_\gamma\Omega\nabla_\sigma\Omega. \tag{A.24}$$

Element of volume:

$$\mathrm{d}^4 v = \sqrt{-g}\ \mathrm{d}^4 x. \tag{A.25}$$

Element $\mathrm{d}\sigma_\alpha$ *of a hypersurface* Σ defined by equations $x^\mu = x^\mu(y^i)$:

$$\mathrm{d}\sigma_\alpha = \frac{1}{3!}e_{\alpha\beta_1\beta_2\beta_3}\det\left(\frac{\partial x^{\beta_i}}{\partial y^j}\right)\mathrm{d}y^1\,\mathrm{d}y^2\,\mathrm{d}y^3, \tag{A.26}$$

where $e_{\alpha\beta\gamma\delta}$ is antisymmetric tensor:

$$e_{\alpha\beta\gamma\delta} = \sqrt{-g}\ \varepsilon_{\alpha\beta\gamma\delta} \tag{A.27}$$

[$\varepsilon_{\alpha\beta\gamma\delta}$ is the completely antisymmetric symbol ($\varepsilon_{0123} = 1$)].

Element $\mathrm{d}\sigma_{\alpha\beta}$ *of a two-dimensional surface* S defined by equations $x^\mu = x^\mu(z^a)$ ($a = 1,2$):

$$\mathrm{d}\sigma_{\alpha\beta} = \frac{1}{4}e_{\alpha\beta\gamma_1\gamma_2}\det\left(\frac{\partial x^{\gamma_a}}{\partial z^b}\right)\mathrm{d}z^1\,\mathrm{d}z^2. \tag{A.28}$$

Integration in the Riemannian space. Let ϕ be a scalar, ϕ^α be a vector, and $\phi^{\alpha\beta}$ be an antisymmetric tensor field. The following integrals are then defined:

$$T_V[\phi] = \int_V \phi\,\mathrm{d}^4 v, \tag{A.29}$$

$$T_\Sigma[\phi^\alpha] = \int_\Sigma \phi^\alpha\,\mathrm{d}\sigma_\alpha, \tag{A.30}$$

$$T_S[\phi^{\alpha\beta}] = \int_S \phi^{\alpha\beta}\,\mathrm{d}\sigma_{\alpha\beta}. \tag{A.31}$$

Stokes' theorems:

$$\int_V \phi^\alpha{}_{;\alpha}\,\mathrm{d}^4 v = \int_{\partial V}\phi^\alpha\,\mathrm{d}\sigma_\alpha, \tag{A.32}$$

$$\int_\Sigma \phi^{\alpha\beta}{}_{;\beta}\,\mathrm{d}\sigma_\alpha = \int_{\partial\Sigma}\phi^{\alpha\beta}\,\mathrm{d}\sigma_{\alpha\beta}, \tag{A.33}$$

where ∂V and $\partial\Sigma$ are the boundaries of the four-volume V and hypersurface Σ.

Induced metric h_{ij} and exterior curvature K_{ij} of a hypersurface. Let $x^\mu = x^\mu(y^i)$ be the equation of a hypersurface Σ, and n^μ be a unit normal to it; let

$$e^\mu_{(i)} = \frac{\partial x^\mu}{\partial y^j} e^j_{(i)}$$

be a triple of mutually orthogonal unit vector tangent to Σ. Then

$$h_{ij} = \frac{\partial x^\mu}{\partial y^i}\frac{\partial x^\nu}{\partial y^j} g_{\mu\nu}, \tag{A.34}$$

$$K_{(i)(j)} = n_\mu e^\nu_{(i)} \nabla_\nu e^\mu_{(j)}, \tag{A.35}$$

where $K_{(i)(j)}$ are the components K_{ij} in the base $e_{(i)}$.

Gauss–Codazzi equations:

$$R^m{}_{ijk} = {}^{(3)}R^m{}_{ijk} + \varepsilon(n)(K_{ij}K^m{}_\kappa - K_{ik}K_j{}^m), \tag{A.36}$$

$$n_u R^\mu{}_{ijk} = -\varepsilon(n)(K_{ij:k} - K_{ik:j}), \tag{A.37}$$

where $\varepsilon(n) = n_\mu n^\mu = \pm 1$, $n_\mu = (\ )_i$ is the covariant derivative in the metric h_{ij}, and ${}^{(3)}R^m{}_{ijk}$ is the curvature tensor of the three-dimensional space with this metric.

'3 + 1' *split of the Einstein tensor*:

$$n^\alpha n^\beta G_{\alpha\beta} = -\frac{1}{2}\,{}^{(3)}R + \frac{1}{2}\varepsilon(n)[K^2 - K_{ij}K^{ij}], \tag{A.38}$$

$$n^\alpha G_{\alpha i} = -\varepsilon(n)[K_i{}^m{}_{:m} - K_{:i}], \tag{A.39}$$

where

$${}^{(3)}R = h^{ik}\,{}^{(3)}R^m{}_{imk},\quad K = h^{ij}K_{ij}.$$

Einstein action:

$$W[g] = \frac{c^3}{16\pi G}\left(\int_V R\sqrt{-g}\ \mathrm{d}^4x - 2\int_{\partial V} K\sqrt{h}\ \mathrm{d}^3y\right). \tag{A.40}$$

Einstein equations:

$$G_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}, \tag{A.41}$$

where $T_{\alpha\beta}$ is the energy-momentum tensor,

$$T_{\alpha\beta}=\frac{-2c\,\delta W_m}{\sqrt{-g}\;\delta g^{\alpha\beta}}, \tag{A.42}$$

and W_m is the action of matter. For the covariant action W_m, $T^{\alpha\beta}{}_{;\beta}=0$.

Energy conditions. Let ξ^μ be an arbitrary time-like vector field.

Weak energy condition signifies that the following inequalities hold for a given $T_{\alpha\beta}$:

$$T_{\alpha\beta}\xi^\alpha\xi^\beta\geq 0. \tag{A.43}$$

Energy dominance condition: $T_{\alpha\beta}\xi^\beta$ is a nonspace-like vector.

Strong energy dominance condition: the inequalities

$$T_{\alpha\beta}\xi^\alpha\xi^\beta\geq\frac{1}{2}\,T^\mu_\mu\xi^\alpha\xi_\alpha \tag{A.44}$$

hold.

Electromagnetic field A_μ.
Action:

$$W[A]=-\frac{1}{16\pi}\int_V F_{\mu\nu}F^{\mu\nu}\sqrt{-g}\;\mathrm{d}^4x+\int_V A_\mu j^\mu\sqrt{-g}\;\mathrm{d}^4x, \tag{A.45}$$

$$F_{\mu\nu}=2A_{[\nu,\mu]}. \tag{A.46}$$

Maxwell's equations:

$$F^{\mu\nu}{}_{;\nu}=4\pi j^\mu. \tag{A.47}$$

Energy-momentum tensor

$$T_{\mu\nu}=\frac{1}{4\pi}\left(F_{\mu\alpha}F_\nu{}^\alpha-\frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right). \tag{A.48}$$

Scalar field ϕ.
Action:

$$W_\xi[\phi]=-\frac{1}{8\pi}\int_V(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}+m^2\phi^2+\xi R\phi^2)\sqrt{-g}\;\mathrm{d}^4x+$$

$$+\int_V J\phi\sqrt{-g}\;\mathrm{d}^4x+\frac{1}{4\pi}\,\xi\int_{\partial V}K\phi^2\sqrt{h}\;\mathrm{d}^3y, \tag{A.49}$$

where m is the field mass, and ξ is a free parameter. If $m=0$ and $\xi=1/6$, the theory is conformally invariant.

Field equations:

$$\Box\phi - (m^2 + \xi R)\phi = -4\pi J. \tag{A.50}$$

Energy-momentum tensor:

$$T^{(\xi)}_{\mu\nu} = \frac{1}{4\pi}\{\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}(\phi_{,\alpha}\phi^{,\alpha} + m^2\phi^2) +$$

$$+ \xi[(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)\phi^2 + g_{\mu\nu}(\phi^2)^{;\alpha}{}_{;\alpha} - (\phi^2)_{;\mu\nu}]\} + J\phi g_{\mu\nu}. \tag{A.51}$$

Conservation laws. Let ξ^μ be a vector field and $\xi^{\mu\nu}$ be a Killing tensor field. If $T^{\mu\nu}$ is a symmetric tensor satisfying the condition $T^{\mu\nu}{}_{;\nu} = 0$ (the energy-momentum tensor), the quantity

$$P_\xi = \int_\Sigma T^{\mu\nu}\xi_\mu \mathrm{d}\sigma_\nu \tag{A.52}$$

is independent of the choice of the total Cauchy surface Σ.

If p^μ is the momentum of a particle ($p^\nu p^\mu{}_{;\nu} = 0$), then the quantities

$$P_\xi = \xi^\mu p_\mu, \quad Q_\xi = \xi^{\mu\nu} p_\mu p_\nu \tag{A.53}$$

are constant along the trajectory of the particle.

The congruence of curves is the three-parameter family of curves $x^\mu(\lambda; y^i)$ (λ is the parameter along the curve and y^i is the parameter that 'enumerates' the curves) such that one and only one curve of the family passes through each point.

If specific λ and y^i are chosen on the congruence, we obtain a coordinate system. A congruence of time-like curves is sometimes known as a reference frame.

Differential invariants of the congruence of timelike curves. Let λ be a proper time parameter and $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda$ be a vector field ($u^\mu u_\mu = -1$) connected with a congruence $x^\mu(\lambda; y^i)$. Then $u_{\alpha;\beta}$ admits an unambiguous representation of the form

$$u_{\alpha;\beta} = \omega_{\alpha\beta} + \sigma_{\alpha\beta} + \frac{1}{3}\theta p_{\alpha\beta} - w_\alpha u_\beta \tag{A.54}$$

where $p_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$ is a projection tensor that projects vectors on three-space orthogonal to u^μ, $w_\alpha = u^\beta u_{\alpha;\beta}$ is the acceleration, $\theta = u^\alpha{}_{;\alpha}$ is the 'divergence' of the world lines of the congruence, $\omega_{\alpha\beta}$ is the rotation tensor, and $\sigma_{\alpha\beta}$ is the shear tensor:

$$\omega_{\alpha\beta} = \omega_{[\alpha\beta]} = \frac{1}{2}(u_{\alpha;\mu}p^\mu{}_\beta - u_{\beta;\mu}p^\mu{}_\alpha), \tag{A.55}$$

$$\sigma_{\alpha\beta} = \sigma_{(\alpha\beta)} = \frac{1}{2}(u_{\alpha;\mu}p^\mu{}_\beta + u_{\beta;\mu}p^\mu{}_\alpha) - \frac{1}{3}\theta p_{\alpha\beta}. \tag{A.56}$$

Raychaudhuri equation:

$$\frac{d\theta}{d\lambda} = w^{\alpha}{}_{;\alpha} + 2(\omega^2 - \sigma^2) - \frac{1}{3}\theta^2 - R_{\alpha\beta}u^{\alpha}u^{\beta}, \tag{A.57}$$

where

$$\omega^2 = \frac{1}{2}\,\omega_{\alpha\beta}\omega^{\alpha\beta},\quad \sigma^2 = \frac{1}{2}\,\sigma_{\alpha\beta}\sigma^{\alpha\beta}.$$

Equation (A.54) is often written in a different form [see Vladimirov (1982)]:

$$u_{\mu;\nu} = A_{\mu\nu} + D_{\mu\nu} + F_{\mu}u_{\nu}. \tag{A.58}$$

The quantities with direct physical meaning are $A_{\mu\nu}$,$D_{\mu\nu}$, and F_{μ} with spatial subscripts, namely: A_{ik} is the tensor of the angular velocity of rotation of the reference frame,

$$A^{ik} = \frac{c}{\sqrt{-g_{00}}}\left(\frac{1}{2}\frac{\partial g^{ik}}{\partial x^0} + g^{i\alpha}\Gamma^{k}_{\alpha 0}\right); \tag{A.59}$$

D_{ik} is the tensor of the rate of shear of the reference frame,

$$D_{ik} = \frac{1}{2}\frac{c}{\sqrt{-g_{00}}}\frac{\partial \overset{*}{h}_{ik}}{\partial x^0}, \tag{A.60}$$

where

$$\overset{*}{h}_{ik} = g_{ik} - \frac{g_{0i}g_{0k}}{g_{00}},\quad \overset{*}{h}_{ik}\overset{*}{h}{}^{il} = \delta^{l}_{k};$$

and F_i is a vector of the field of gravitational-inertial forces that act in the frame of reference (i.e., the vector of the free fall acceleration of a test particle at rest),

$$F^i = c^2\frac{\Gamma^{i}_{00}}{g_{00}} \tag{A.61}$$

The vector of the angular velocity of rotation of the reference frame, Ω_i is calculated using A_{ik}:

$$\Omega_i = \frac{1}{2}\varepsilon_{ikl}A^{kl}, \tag{A.62}$$

where ε_{ijk} is the absolutely antisymmetric object, $\varepsilon_{123} = (g/g_{00})^{1/2}$.

The scalar $\Omega = \sqrt{(\Omega_i\,\Omega_k h^{*ik})}$ is the angular velocity of rotation per unit of proper time $d\tau = \sqrt{-g_{00}}\;dt$.

The scalar

$$F = \sqrt{F^i F^k \overset{*}{h}_{ik}} \tag{A.63}$$

is the magnitude of free-fall acceleration of a body at rest in the chosen reference frame.

Notes

Preface

[1] Some aspects of the physics of black holes were treated, for instance, in the following review papers: Markov (1970, 1973), Penrose (1972), Carter (1973a), De Witt (1975), Sexl (1975), Zel'dovich *et al.* (1976), Sciama (1976), Frolov (1976b, 1978b, 1983b), Ruffini (1979), Bekenstein (1980), Israel (1983), and Dymnikova (1986).

Chapter 1

[1] On the black-hole problem in non-Einsteinian theories of gravitation, see, e.g., Will (1981).

Chapter 2

[1] Strictly speaking, a general-type metric is described by (2.1.6). Formula (2.1.2) holds everywhere except on special surfaces where

$$\frac{\partial \tilde{g}_{22}}{\partial \tilde{x}^0} = \frac{\partial \tilde{g}_{22}}{\partial \tilde{x}^1} = 0.$$

[2] Expression (2.2.6) gives acceleration, and $\mathfrak{F}_0 = Fm$ gives the force acting on a body of mass m, measured by an observer located near this body at a point r_0. If the body is suspended on a weightless absolutely rigid string, then the force applied to the free end of the string at the point r_1 is

$$\mathfrak{F}_1 = \mathfrak{F}_0 \sqrt{\frac{g_{00}(r_0)}{g_{00}(r_1)}}.$$

As r_0 tends to $2GM/c^2, \mathfrak{F}_0 \to \infty$, while $\mathfrak{F}_1$ remains finite.

[3] These questions are discussed in detail in Section 4.2.

[4] Recall that in the general case of nonspherical time-dependent gravitational fields, it is not possible to introduce a rigid three-dimensional space; this fact stands in the way of clear intuitive concepts and inhibits calculations.

[5] The fact that the lines r = const are plotted in R,T coordinates by straight lines is a corollary of the special choice of R [see (2.4.2)]. This is one of the reasons why we chose the coordinate R instead of r_1.

[6] One never should forget the modification in the meaning of the coordinates r and t when $r < r_g$ (see above).

[7] Discontinuities in g_{22} result in discontinuities in spacetime.

[8] If the orbit as a whole lies far from the black hole, it is an ellipse which slowly rotates in the plane of motion.

[9] Of course, a particle may escape from a white hole and fall into a black one; this case was discussed in Section 2.7.

[10] The process of radiation in the theory of the strong field of the black hole is treated in Chapter 3.

Chapter 3

[1] Here we do not treat nonphysical perturbations due to a small displacement of a black hole as a whole in the given reference frame.

Chapter 4

[1] It will be clear hereafter that the black hole forms only if the inequality $M^2 \geq J^2/M^2 + Q^2$ is satisfied (the system of units is such that $G = c = 1$).

[2] At large distances from the black hole, this reference frame turns into the Lorentz frame in which the black hole is at rest.

[3] We have chosen the Killing vector field in such a way that far from the black hole (at infinity) the Killing vector is directed along the time lines of the Lorentz coordinate system into which the system (4.2.1) converts. This clarification is necessary because in addition to stationarity, metric (4.2.1) has the property of axial symmetry (is independent of angle ϕ). Hence, there is another Killing vector field due to the invariance of space under rotations around the symmetry axis. A linear combination of two Killing vectors is always a Killing vector (i.e., in our case we can combine a shift of section in time with a rotation around a spatial axis). We single out a Killing vector that corresponds to a lack of any rotation of the space grid around the symmetry axis at large distances from the black hole ($r \to \infty$).

[4] Another approach to the '3 + 1' split is possible, where one chooses first not three-dimensional sections but congruences of time-like lines. This approach is known as the chronometric method [Zel'manov (1956)].

[5] $\hat{F}_{\hat{\imath}}$ are the components of the acceleration vector, measured directly by an observer at rest in the given reference frame. In formula (4.3.2), they are given in a local Cartesian coordinate system whose axes are aligned along the directions of r, θ, and ϕ.

[6] Of course, the coordinate grid can be extended closer to the black hole but it cannot be constructed of material bodies.

[7] Several remarks on terminology. In fact, it is not unified among different authors and even in different papers of one author. Thus, MacDonald and Thome (1982) refer to the space that we describe in Section 4.2 as 'absolutely rigid' and specify that locally nonrotating observers move in this space [see Section 2 of their paper prior to formula (2.6)]. In another paper, Thome (1985) refers to the space comoving with locally nonrotating observers as absolute, and says that the observers are at rest in the 'absolute' space [see Thome (1985), p. 11; see also; Thome, Price, and MacDonald (1986)]. We invariably hold to the former viewpoint.

We have mentioned that several terms are used to indicate the chronometric reference frame (see p. 54). This 'discord' has historic roots, but the authors hope that certain unification will be established soon.

[8] Of course, we ignore the trivial coordinate singularity at the pole of the spherical coordinate system: Everyone is used to it and its meaning is obvious.

[9] The structure of the maximal analytic continuation of the Kerr–Newman metric is discussed in Section 6.5.

[10] This integral of motion is implied by the existence of the Killing tensor (A.17) field in the Kerr metric [Carter (1968a, 1973a, 1977), Walker and Penrose (1970)].

[11] It is important that these calculations ignore the back-reaction of radiation on the motion of the particle, and thus cease to hold when $\tilde{L}$ is close to the critical values determining the capture of a particle. At such values of $\tilde{L}$ the particle goes through many revolutions close to the critical orbit (see Sections 4.5 and 4.6); the back-reaction of radiation on the motion accumulates and has to be taken into account. The case of $|\tilde{L}|>L_{cr}$ is analyzed later in the Chapter.

[12] The sign of m is chosen in accord with Figure 34 and Detweiler's paper (1980); Kojima and Nakamura use the opposite convention.

Chapter 5

[1] The metric $d\tilde{s}^2$ has removable coordinate singularities at $\xi = 0$ and $\xi = \pi$.

[2] M is referred to as the Penrose space.

[3] For details on congruences on null geodesics in curved spacetime, see, e.g., the reviews by Pirani (1964) and Frolov (1976a) where the relevant references are given.

[4] Optical scalars ρ and σ are defined by (5.3.19) for an arbitrary null geodesic congruence; relations (5.3.22) and (5.3.23) are then also satisfied. If l^{μ} are tangent vectors to the null surface, then ρ satisfies the condition $\rho = \bar{\rho}$.

[5] The cosmic censor conjecture in this formulation is sometimes referred to as 'weak', to distinguish it from the 'strong' one. The latter principle, suggested by Penrose (1978), states

that in the general case the singularities produced by the gravitational collapse are space-like so that no observer can see them until he falls into them.

6 For the discussion of singularities and references to relevant publications, see Section 5.6.

7 Possible violations of the Hawking theorem in quantum creation of particles in the field of a black hole were first pointed out by Markov (1974).

8 Note that after the star has contracted to r_g (after τ_1), the explosion cannot eject the matter out to infinity. Nothing can escape from under $r = r_g$.

9 In proving rigorous results concerning the properties of black holes, one usually has to make certain assumptions on the global properties of spacetime. These assumptions are reasonable in the physics context and play an important 'technical' role in the proofs of theorems, since they help to ignore various 'pathological' situations. Such assumptions are usually listed in detail when the relevant theorems are discussed [see Hawking and Ellis (1973)]. We made an attempt to restrict as much as possible the use of numerous terms meant to denote these properties. Suffice it to mention that a partial Cauchy surface is a space-like surface intersected by each causal curve not more than once. By definition, *a space with strongly asymptotically predicable future* is the space with such partial Cauchy surface that not only the 'exterior' of the black hole but also a small neighborhood of the event horizon become predictable by fixing initial data on this surface. We also use the concept of *regularly predictable space*. This is a space with a strongly asymptotically predictable future and the partial Cauchy surface Σ with the following properties: (1) the intersection of Σ and $J^-(\mathcal{J}^+)$ is contained in $J^+(\mathcal{J}^-)$ and is homeomorphic to R^3 from which an open set with compact closure is cut out, and (2) the surface Σ is simply connected.

10 The region $D^+(\Sigma)$ of the Cauchy future of set Σ is defined as the set of all points p such that each causal curve nonextendable into the past and passing through p necessarily intersects Σ.

11 It must be emphasized that closed time-like lines do not contradict the broadly interpreted causality principle. The future cannot be distinguished from the past on a closed time line but this property, as such, does not produce any contradictions [see Zel'dovich and Novikov (1975), Novikov (1983)]. All events on such a line are 'synchronized' with one another. The past cannot be changed once the future is known, (as it is sometimes said) because all the events along the time line 'have already occurred'; they cannot be altered, since they are a part of the four-dimensional spacetime. A different wording is possible: It is wrong to speak of the effect of the future on the past because these concepts become meaningless. For more details, see Novikov and Thome (1988).

Chapter 6

1 The condition $T_t^t = T_t^r$ can be shown to be necessary and sufficient for Birkhoff's theorem to hold (this theorem guarantees the staticity of spherically symmetric solutions). Obviously, this condition holds in the Reissner–Nordström and de Sitter spaces.

2 If magnetic monopoles exist and the collapsing system possesses a magnetic charge, the value of this charge must be introduced into the description of the black hole as the fourth parameter.

3 The conservation law for ε ($p^\mu \varepsilon_{,\mu} = 0$) follows from the relation $p^\nu \varepsilon_{,\nu} = p^\nu p^\mu{}_{;\nu} \xi_\mu + p^\mu p^\nu \xi_{\mu;\nu}$, from the geodesic character of the motion $p^\nu p^\mu{}_{;\nu} = 0$, and from Killing equation (6.2.1).

4 On the analyticity of stationary axisymmetric asymptotically flat solutions of Einstein's equations, see Müller zum Hagen (1970).

5 Action (6.4.15)–(6.4.16) is a particular case of action of the type

$$S[Z^A] = \int dx \sqrt{\gamma}\, \gamma^{ab}\, \partial_a Z^A\, \partial_b Z^A G_{AB},$$

where $a,b = 1,\dots, n$; $A,B = 1,\dots, N$; $\gamma^{ab} = \gamma^{ab}(x)$; $G_{AB} = G_{AB}(Z)$. The extremal of this action, $Z^A(x)$ is said to be the harmonic mapping [Misner (1978)]. The invariance mentioned above signifies that in the case considered here, the nonphysical space [Z^A are its coordinates and $G_{AB}(Z)$ is its metric] is homogeneous. Bunting (1983) suggested a different proof of the uniqueness theorem. His proof does not make use of this symmetry but is based on the fact that the curvature tensor

of this space is sign-constant. For the exposition of this proof and its possible generalization, see Carter (1985).

[6] Even though we have assumed in this section that the black hole has no magnetic charge, the results presented here are readily generalized to the case when the charge does not vanish.

[7] For metric (4.2.1) to be asymptotically flat at infinity, $r \to \infty$, the variable ϕ must vary from 0 to 2π, and θ from 0 to π.

[8] The structure of the maximal analytic continuation for an extremal black hole is somewhat different [see Carter (1966a), Hawking and Ellis (1973)]. The maximal analytic continuation for the Reissner–Nordström metric was obtained by Graves and Brill (1960). The general method of constructing a maximal analytic continuation for stationary metrics with horizons was presented by Walker (1970).

[9] It is possible to get into the region V' directly from II across the intersection of the boundaries r_-.

[10] Note that if the condition of finiteness of ϕ is dropped, it is possible to construct a solution that describes an extremal nonrotating black hole with scalar massless field [Bocharova *et al.* (1970), Bekenstein (1975)]. This solution was found to be unstable [Bronnikov and Kireev (1978)].

[11] On black hole solutions in theories of gravitation with higher-in-curvature corrections, see Wiltshire (1986, 1988), Myers (1987), Whitt (1988), and references cited therein.

Chapter 7

[1] On the properties of electromagnetic fields inside the black hole, see Section 12.1.

[2] The slow change in the parameters of the black hole as a result of electromagnetic processes is analyzed later (Section 7.4).

[3] In view of the applications where the electrodynamic formulas of this chapter are used in astrophysics, we write them in a conventional system of units, retaining c.

[4] The case of nonstationary field is discussed, e.g., by MacDonald and Suen (1985), Thorne (1986).

[5] Note that the metric distance from any point in the section t = const to the event horizon is nevertheless finite (excepting the case of $a = M$). This effect occurs because the null geodesic tends to align parallel to this section as the horizon approaches.

[6] If we calculate the components of the field $E_\perp$ and $B_\perp$ at the event horizon (see Section 7.3), we find that both are proportional to $r_+ - M$, where $r_+ = M + (M^2 - a^2)^{1/2}$. Hence, we obtain for a black hole rotating at maximum angular velocity, with $a_{\max} = M$, that $E_\perp = 0$ and $B_\perp = 0$ at the horizon, that is, the lines of force of the axisymmetric field do not pass through the black hole (the case of asymmetric field is treated later). Problems in black-hole electrodynamics were also treated by Lèaute and Linet (1976, 1978), Misra (1977), Gal'tsov and Petukhov (1978), Linet (1976, 1977a,b, 1979), Demianski and Novikov (1982), Bicak and Dvorak (1980); see also Sections 8.2-8.4.

[7] Of course, the total flux of magnetic field across any closed surface, including the horizon, equals zero (see Section 7.1). In speaking about the flux through a black hole, one considers the incoming (or the outgoing) lines of force.

[8] Obviously, the total charge of the black hole in the stationary solution is always zero because the total numbers of charges of opposite signs, sinking into the black hole, are equal to each other.

[9] A similar analysis of slipping of particles along the magnetic lines of force at the black hole horizon shows that the condition $|v_{\min}| \approx c$ corresponds to boundary conditions (7.3.11) and (7.3.13).

Chapter 8

[1] Specific features of physical processes in the field of a black hole, for which quantum phenomena are important, are discussed in subsequent chapters.

[2] Electromagnetic and gravitational waves propagating close to a charged black hole may transform into each other (for details, see Section 8.4). This mixing does not change the general condition of wave amplification but calls for a more careful analysis.

[3] Note that this formulation of the problem is, to a certain extent, idealized because the instability inherent in such a black hole is ignored (see Section 12.2 and 13.2).

[4] Strictly speaking, Equation (8.2.2a) is only valid for the total electric field (i.e., for the sum of the fields of test charge and black hole). If the field of the test charge is treated as perturbation, the equation for this perturbation differs from (8.2.2a) in the terms describing the perturbation of the gravitational field (see Section 8.4). Following Linet and Leaute, we drop these additional terms. This operation does not change the general conclusion on the specific features of the field inside the black hole. These additional terms disappear if $Q = 0$.

[5] A discussion of the boundary conditions for the field of a test charge in the spacetime of an eternal black hole can be found in Demianski and Novikov (1982).

[6] On the behavior of a massive vector field in the neighborhood of a Schwarzschild black hole, see also Gal'tsov *et al.* (1984).

[7] We use here the definition (A.26) for $d\sigma_\mu$ in which an element of volume of the hypersurface $x^0 = \text{const}$ equals $d\sigma_\mu = \delta^0_\mu \sqrt{-g}\, d^3x$.

[8] Expressions (8.3.4)–(8.3.5) for the self-energy of a charge, together with the above condition imposed on the choice of integration domain Σ, are in complete agreement with general expression (11.2.48) for the change in the mass of a system, containing the black hole, due to a change in the parameters of the entire system. It is interesting to note that the integral over the part of the surface Σ under the horizon is identically zero for the Schwarzschild black hole.

[9] On the application of the geometrical optics approach to the propagation of high-frequency gravitational waves, see Isaacson (1968a, b). A detailed presentation of this approach for electromagnetic and gravitational perturbations can be found in Misner, Thorne, and Wheeler (1973).

[10] Obviously, the observer falling into the hole will find that the light wavefront takes a finite time (by his clock) to cross the event horizon.

[11] In general, the inhomogeneities of the gravitational field are characterized both by the curvature radius L_1 of spacetime and by the scales of the space (L_2) and time (L_3/c) over which this curvature changes appreciably. The minimal among these quantities is denoted by L: $L = \min(L_1, L_2, L_3)$.

[12] On the effects connected with the black hole reaction to electromagnetic fields, see also Gal'tsov *et al.* (1984).

[13] This property of the Schwarzschild metric was discovered by Pirani (1959). Later Penrose (1976) proved that general-type spacetime has this property. Namely, the geometry of the surrounding spacetime tends to that of a flat gravitational wave for an observer whose velocity tends to the speed of light and who uses the time parameter $\gamma\tau$ where $\gamma = (1-v^2)^{-1/2}$ and τ is the proper time. In the limit $\gamma \to \infty$, the world line of this observer is a null geodesic and $\gamma\tau$ is the affine parameter along this geodesic.

[14] Strictly speaking, what is needed is the regularity of the so-called Killing horizon that is defined on a static space by the condition $\partial_t \cdot \partial_t = 0$, where ∂_t is the Killing vector field. The Killing horizon coincides with the event horizon in static asymptotically flat spaces (see Section 6.3).

[15] It is of interest to note that even though the Weyl metric describing a distorted axially symmetric black hole belongs to the Petrov type I, its type on the event horizon is D [Papadopoulos and Xanthopoulos (1984)].

[16] It should be mentioned that vacuum axisymmetric static metrics are possible, describing the spacetime close to the horizon which has the topology of torus [see, e.g., Geroch and Hartle (1982), Xanthopoulos (1983), and references cited in these papers]. However, the Hawking theorem implies (see Section 6.2) that if the energy dominance condition is satisfied, the external space cannot be regular and at the same time asymptotically flat.

Chapter 9

[1] Note that the relations based on dimensional analysis, (9.1.3)–(9.1.6), give a rough estimate only. The properties of the radiation of rotating black holes are discussed in Section 9.5. Suffice it to remark here that the intensity of this radiation essentially depends on the particle spin. The

values of dE/dt and dJ/dt for gravitons, for $J = GM^2/c$, are less than those given by (9.1.5)–(9.1.6) by an order of magnitude, and those for neutrinos, by three orders of magnitude.

2 Strictly speaking, the radiation spectrum of a black hole deviates from the blackbody spectrum owing to the effects of scattering on the gravitational field (see Section 9.5 and Figure 78).

3 A similar estimate can be obtained if we take into account the quantum nature of motion of the particle falling into the black hole or if we try to evaluate the accuracy with which the position of the falling body can be localized in the vicinity of the event horizon using the method of scattering of the waves of a physical field on this body. In the latter case, the restriction (9.1.12) arises because the size δr of the wave packer, whose energy is less than the black hole mass M, must be greater than the quantity $\delta r \sim h/Mc$.

4 A more detailed presentation of this theory, and also of the theory of Fermi fields in curved spacetime, can be found in De Witt (1965, 1975), Birrel and Davies (1982).

5 If the space is non-compact, the solutions ϕ^1 and ϕ^2 are assumed to fall off sufficiently rapidly at infinity.

6 Note that the commutation relations written in the form (9.2.6) have a larger range of applicability than (9.2.5). Thus, they are valid for the theory with constraints [i.e., when $\det(P^{AB00}) = 0$] that dictate a change in rules (9.2.5) of standard canonical quantization.

7 As a result of this operation, all creation operators in the expansion of the appropriate expression into a series in creation and annihilation operators are placed on the left of annihilation operators. The same expression (9.2.29) describes the S-matrix in the case of Fermi fields as well. The matrices Λ and V are then antisymmetric. The general expression for them in terms of Bogoliubov coefficients is given in the monographs of Berezin (1965) and De Witt (1965).

8 Unruh (1981) pointed out that the phenomenon of quantum creation of particles in black holes permits a hydrodynamic analogy. If the flow of the fluid is such that a closed two-dimensional surface separates the region of subsonic and supersonic flow (Laval nozzle), the system must radiate thermal-spectrum phonons.

9 Note that if the size of the radiation-filled cavity is sufficiently large, this equilibrium is generally unstable (for details, see Section 11.4).

10 Provided the conditions that guarantee the absence of superradiative modes are met.

Chapter 10

1 When the Feynman diagram technique is used in quantum theory for calculating the indicated expectation values, the expansion in $\hbar$ coincides with the expansion in closed loops found in the corresponding diagrams. Terms of order $\hbar^0$ are described by diagrams without loops (the 'tree' approximation), and those of order $\hbar^1$, by diagrams with a single loop ('one-loop' approximation).

2 The values of the coefficients a_s and b_s in (10.1.4) were obtained by Duff (1977) by the method of dimensional regularization. Additional terms of type $\Box R$ may appear if other methods are employed in the case of the electromagnetic field. Note that this ambiguity vanishes in metrics satisfying the vacuum Einstein's equations ($R_{\mu\nu} = 0$), so that we obtain,

$$\langle T^{\mu}_{\mu}\rangle^{\text{ren}} = (a_s + b_s) C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta}.$$

The values of the coefficients a_s and b_s for the four-component massless Fermion field are twice those given in (10.1.4).

3 If the points x and x' are separated by a space-like distance, then $\hat{\varphi}_A(x)$ and $\hat{\varphi}_{B'}(x')$ commute and $G^{(1)}_{AB'}(x, x') = -2iG_{AB'}(x, x')$.

4 A general analysis of the problem of defining the vacuum in a spacetime with horizons can be found in the papers of Fulling (1977a, b), Sciama *et al.* (1981) Fulling and Ruijsenaars (1987).

5 Rather, it is nearly stationary because its parameters change to some extent as a result of quantum evaporation. But we have already mentioned that the rate of this change is negligibly small only as long as the mass of the black hole is much greater than the Planck mass.

[6] Note that in the Riemannian space, Green's function of a massive field for a metric with Euclidean signature falls off exponentially as the distance between the points x and x' increases. If these points are far from the boundaries, the effect of boundary conditions is indeed negligible for these fields.

[7] In these formulas (and those that follow) the superscript 'ren' is dropped in $\langle\phi^2\rangle^{\text{ren}}$ and $\langle T^{\mu}_{\nu}\rangle^{\text{ren}}$ because we will operate only with renormalized values of these quantities. The superscript after the angle brackets indicates the vacuum state under which the averaging is done. Thus $\langle\phi^2\rangle_B$ denotes $\langle B|\phi^2|B\rangle^{\text{ren}}$.

[8] The cases where the point of interest lies on the horizon belong to important exceptions in which exact values of $\langle\phi^2\rangle^{\text{ren}}$ and $\langle T_{\mu\nu}\rangle^{\text{ren}}$ can be obtained. See later.

[9] It is interesting to note that the components of $\langle T^{\mu}_{\nu}\rangle_U$ describing the energy and angular momentum fluxes across the horizon of a rotating black hole can be calculated using a simple approach based on the equivalence principle [Frolov and Thorne (1989)]. For a general discussion of the equivalence principle in quantum domain, see, e.g., Ginzburg and Frolov (1987).

[10] Likewise, an observer moving at an acceleration a in a flat spacetime will also record a temperature $T_a = a/2\pi$ using the thermometers described above. From the standpoint of this observer, the standard Minkowski vacuum behaves, in a sense, just as the thermal radiation at a temperature T_a. Note that while recording the energy of these thermal-energy particles, the observer cannot measure their momentum with sufficient accuracy because the characteristic wavelength of this 'radiation' is of the order of distance to the horizon. The same remark holds for the 'particle' of the thermal radiation recorded by the observer near the black hole. For details, see Unruh (1976b), Unruh and Wald (1984).

[11] It must be mentioned in this connection that the conformal anomaly in two-dimensional space is of the form

$$\langle T^{\mu}_{\mu}\rangle = \frac{C_s}{24\pi} R,$$

where R is the scalar curvature and C_s is a coefficient dependent on the spin s of the field ($C_0 = 1$). The total energy-momentum tensor $\langle T^{\mu}_{\nu}\rangle$ is determined by its trace up to two functions of one variable, that correspond to the arbitrariness in the choice of the boundary conditions [Christensen and Fulling (1977)]. As a result, it is possible to calculate $\langle T^{\mu}_{\nu}\rangle$ exactly in a number of two-dimensional models imitating the black hole [see Davis (1976), Davis *et al.* (1976), Unruh (1977), Fulling (1977b), Hiscock (1977), Frolov and Vilkovisky (1983), Balbinot and Brown (1984), Balbinot (1984a), Kuroda (1984a), Birrel and Davies (1982)].

[12] A discussion of the general properties of surfaces formed by stable points under the action of a symmetry group can be found in Boyer (1969); see also Miller (1979).

[13] This approach can be used to calculate $\langle\phi^2\rangle_H$ for a black hole placed in external axially symmetric static gravitational field [Frolov and Garcia (1983)] or enclosed in a mirror-walled cavity described by the equation $r = r_0$ ($\phi|_{r_0} = 0$). In the latter case, the problem reduces to calculating the field produced inside a conducting grounded ellipsoid of revolution,

$$\frac{R^2}{\Delta(r_0)} + \frac{z^2}{(r_0 - M)^2} = 1,$$

by a pointlike charge placed at its focus. The corresponding value of $\langle\phi^2\rangle_{H;r_0}$ is

$$\langle\phi^2\rangle_{H;r_0} = \langle\phi^2\rangle_H - \delta_{r_0}\langle\phi^2\rangle,$$

where

$$\delta_{r_0}\langle\phi^2\rangle = \frac{1}{8\pi^2(r_+^2 + a^2)} \sum_{l=0}^{\infty} (2l+1)\, \frac{Q_l(l)}{P_l(b)}.$$

Here $b = (r_0 - M)/(M^2 - a^2)^{1/2}$, and P_l and Q_l are Legandre functions.

[14] The values of $\langle T^{\mu}_{\nu}\rangle$ in different 'vacua' differ by the contribution of real thermal radiation particles; hence, in the vicinity of a black hole, $\langle T^{\mu}_{\nu}\rangle_B$ and $\langle T^{\mu}_{\nu}\rangle_U$ practically coincide with $\langle T^{\mu}_{\nu}\rangle_H$ everywhere, except the exponentially narrow strip [$\sim r_g \exp(-\beta GMm/\hbar c)$] near the event horizon. The behavior of these quantities inside the strip is not identical: $\langle T^{\mu}_{\nu}\rangle_B$ diverges on H^- and H^+, while $\langle T^{\mu}_{\nu}\rangle_U$ diverges on H^- but remains finite on H^+.

Chapter 11

[1] The existence of Killing vector field (11.2.9) having this property is implied by the Hawking theorem (Section 6.2) which also implies that Ω^H = const.

[2] When a charged rotating black hole is described in the framework of a five-dimensional Kaluza–Klein theory of gravitation, the quantities Ω^H and Φ^H enter the expressions in a similar manner and their properties are to a certain extent similar [Bleyer *et al.* (1987)].

[3] Direct checking will demonstrate that formulas (11.2.34) and (11.2.37) hold for the mass M and angular momentum J if one takes into account that the metric far from the rotating body can be written in the following form:

$$ds^2 = -\left[1 - \frac{2M}{r} + O(r^{-2})\right]dt^2 - \left[\frac{4J}{r}\sin^2\theta + O(r^{-2})\right] dt\, d\phi +$$

$$+ (1 + O(r^{-1}))[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)].$$

[4] It must be emphasized that the expression (A.28) which we use for the surface element $d\sigma_{\mu\nu}$ agrees with that employed by Bardeen, Carter, and Hawking (1973) and Carter (1973a), but is twice as small as that in Carter (1979) and Damour (1982). Note also that the orientation of $d\sigma_{\mu\nu}$ on the surfaces $\partial\Sigma_\infty$ and $\partial\mathfrak{B}$ was chosen so as to have

$$\int_{\partial\Sigma} \phi^{\mu\nu} d\sigma_{\mu\nu} = \int_{\partial\Sigma_\infty} \phi^{\mu\nu} d\sigma_{\mu\nu} - \int_{\partial\mathfrak{B}} \phi^{\mu\nu} d\sigma_{\mu\nu}.$$

[5] The existence of a gauge in which these relations are satisfied can be verified if one uses the conditions $\mathcal{L}_{\xi(t)}F_{\mu\nu} = \mathcal{L}_{\xi(\phi)}F_{\mu\nu} = 0$ of stationarity and axial symmetry of the field $F_{\mu\nu}$.

[6] The generalized second law in this formulation was first suggested by Bekenstein (1973b, 1974) before the quantum radiation of black holes was discovered.

[7] On the validity of this constraint for real physical systems, see Bekenstein (1982, 1983), Unruh and Wald (1983a).

[8] For a scalar massless field in a two-dimensional space-time this flux, $T_{\mu\nu}v^{\mu}(a^{\nu}/a)$ (v^{μ} is the four-velocity, a^{μ} is the four-acceleration), equals $1/12\pi\, da/d\tau$ (τ is the proper time of the observer on the surface of the mirror).

[9] It must be specially emphasized that another formulation of the third law of thermodynamics, stating that the entropy of a system vanishes at the zero absolute temperature, is not valid for black holes because the area A remains finite as $\kappa \to 0$.

[10] Moss (1985) pointed out that when a black hole undergoes quantum evaporation, a 'bubble' of the new phase may form around it as a result of the phase transition in the surrounding space. Under certain conditions, the particles emitted by the black hole are reflected at the bubble walls, accumulate inside it, and produce a high-temperature medium outside the black hole [see also Hiscock (1987)].

[11] Gibbons and Perry (1976, 1978) demonstrated that this conclusion remains unaltered when the interaction among thermal quanta is taken into account.

[12] A black hole cannot be in a stable thermal equilibrium with an infinitely large reservoir of thermal energy. Hence, the ordinary canonical ensemble of statistical mechanics cannot be used for the description of systems incorporating black holes. However, the description in terms of the microcanonical ensemble remains possible [Hawking (1976a)].

Chapter 12

[1] Sufficiently small $\tilde{\nu}$ are of no interest to us when a 'noneternal' black hole is considered, that is, a black hole created by collapse.

[2] Here we consider black holes with $r_+ > \lambda$ where $\lambda = \hbar/mc$ is the Compton length of the electron [for the analysis of the opposite case, see Page (1977)].

Chapter 13

[1] On the possibility of the creation of a black hole in phase transitions in the early Universe, see Sato *et al.* (1981), Maeda *et al.* (1982), Kardashev and Novikov (1982).

[2] The theorem on the 'falling of hair' near the singularity inside the black hole (see Section 12.1) states that the farther we are from a collapsing nonrotating body, the less the deviation of the spacetime in the T_- -region is from spherical symmetry. Hence, this theorem gives ground to assuming that the conclusions made for spherically symmetric black holes may prove to be valid for more general situations.

[3] For detailed study of the geometry of evaporating black holes, see also Volovich *et al.* (1976), Hiscock (1981), Balbinot and Bergamini (1982), Balbinot *et al.* (1982), Balbinot (1984a), Kuroda (1984).

[4] Recall that owing to quantum effects, the total effective energy-momentum tensor in Einstein's equation does not satisfy, in the general case, the conditions of positive energy density and positive pressure. As a result, quantum effects may violate the conditions of existence of a singularity inside a black hole (see Section 5.6), so that the singularity may be absent.

[5] Zel'dovich (1976) pointed out that if the Hawking radiation creates heavy particles whose decay violates the CP parity, an excess of the baryon or antibaryon charge may arise. These processes were analyzed in detail by Dolgov (1980a, b, 1981). These processes are essential only at a relatively late stage of evaporation, when the black hole temperature reaches a value $\theta = 1/8\pi M \sim 10^{14}$ - 10^{15} GeV; hence, the baryon asymmetry of the decay cannot substantially change the baryon charge of the black hole of mass $M \gg$ 1g, consisting of baryons [see the review of Dolgov and Zel'dovich (1980, 1981)].

[6] These processes were analyzed in detail in connection with the origin of the baryon asymmetry of the Universe [see, e.g., Dolgov and Zel'dovich (1980, 1981), Barrow (1983), Kolb and Turner (1983)]. Estimates of the rate of neutralization of the baryon charge in the superdense matter in Grand Unification theories can be found in Fry (1980a, b, c) and Kolb and Turner (1983).

[7] One should keep in mind that the purely quantum-gravity mechanism of baryon charge nonconservation, suggested by Hawking (1984), may prove to be important at Planck densities.

[8] We emphasize that this conclusion is obtained without taking into account the gravitational field fluctuations. On the possible relationship between the fluctuations of the apparent horizon and quantum radiation of black holes, see Kodama (1980).

[9] On the mechanism of coherence loss in the scattering by quantum-gravity fluctuations, see Hawking (1984, 1988), Lavrelashvili *et al.* (1988)

[10] Note that the existence of heavy magnetic monopoles in nature, as predicted by Grand Unification theories, could have a curious consequence for small black holes [Gibbons (1977), Hiscock (1983)]. The extremal (with a magnetic charge) black holes of mass $M > 150 \times (10^{17}\,\mathrm{GeV}/\mu)^2 g$ (μ is the monopole mass in GeV) would have a lifetime greater than that of the Universe because the Hawing temperature of such a hole is zero and the process of monopole creation is suppressed owing to their high mass.

[11] Note that charged elementary black holes are of great interest for analyzing the problem of self-energy of charged particles. In the framework of classical theory, the gravitational mass

defect creates a difference between the mass M observed at infinity and the internal mass M_0 of the system. If the system is neutral, configurations are possible at a fixed value of M_0 at which M is arbitrarily small [Zel'dovich (1962a)] or identically vanishes (e.g., the case of M_0 forming a closed world). In charged systems (with charge Q), the value of M is bounded from below by the value $Q/\sqrt{G}$ (for the magnetic charge, by $P/\sqrt{G}$) [Arnowitt *et al.* (1963), Markov and Frolov (1970, 1972), Gibbons and Hull (1982), Ludvigsen and Vickers (1983)]. The mass $e/\sqrt{G}$ for the electron is 1.86×10^{-6} g, less by an order of magnitude [by a factor $(\hbar c/e^2)^{1/2} \approx 11.7$] than the Planck mass. Such classical solutions describing charged elementary black holes were given the name 'friedmons'. Their properties are discussed in detail by Markov and Frolov (1970, 1972).

[12] Strong restrictions on the admissible mean density of maximons in the Universe can be obtained using arguments similar to those employed in the derivation of the restrictions on the number of the monopoles and other massive particles [see, e.g., Polnarev and Khlopov (1985)].

[13] Note that in this situation the Planck length acts as a sort of fundamental length. It can be shown [Ginzburg and Frolov (1976)] that in the most general case, the introduction of a fundamental length yields a restriction from below on the spectrum of possible black holes.

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